1

DIMENSIONS AND VECTOR ANALYSIS

1.1 INTRODUCTION

As we are aware that the basic purpose of all sciences is to understand the natural phenomena that occur around us. Amongst all the branches of science, physics is one of the most fundamental. It is the foundation on which the other physical sciences such as chemistry, geology, geophysics, astronomy etc. are based. Physics also plays a very important role in the development of biological sciences.

In physics we perform experiments, make measurements and then propose theories which predict the results of measurements. In this lesson you will first learn about the units of measurements. Every unit of measurement can be expressed in terms of the basic units. This will lead us to the concept of dimensions and their applications in other areas of physics.

We will also catagorise the physical quantities in two groups namely (i) scalars and (ii) vectors depending upon their nature. Finally, we will learn the simple mathematical operations associated with scalars and vectors. You will find application of vectors in different fields of physics which you will learn in other lessons during your course of study.

1.2 OBJECTIVES

After studying this lesson, you should be able to,

- *distinguish between the fundamental and derived quantities and their SI units;*
- *write the dimensions of different physical quantities;*
- *apply the dimensional equations;*
- *differentiate between scaler and vector quantities with examples;*
- *find the resultant of two vectors, and resolve a vector into its components; and*
- *compute the product of two vectors.*

1.3 UNITS OF MEASUREMENTS

The laws of physics are defined in terms of physical quantities like distance speed, time force, area, volume etc. These quantities in turn are defined in terms of more basic quantities like mass, length and time and some others which we will study later.

If a person measures the quantity of milk, she should express the volume of milk in some accepted units of volume. Like-wise if an engineer measures the length of a road connecting two cities, he should express the distance in an accepted unit of length. Such a procedure makes the life more comfortable. When we travel we have an estimate of distance and time which helps us proper planning of the journey. If there were no units accepted by all, the life would be miserable. Such units are much more essential in scientific measurements to facilitate communication of information at international level.

1.3.1 The SI Units

Keeping this point of view in mind, there have been attempts over centuries in several developed civilizations to suggest standard units of measurements at regional or national level. Without going into the long history of the various stages of development in the system of units of measurements, we come to the year 1967 when the XIII General Conference on Weights and Measures, rationalised the MKSA (Metre, Kilogram, Second, Ampere) system of units and adopted a system based on six basic units. It was called the "System International units" known as SI system of units in all languages. In 1971 the General Conference added another basic unit 'mole' for the amount of substance to the SI The present SI system of units has ***seven base or fundamental*** units. These are listed in Table 1.1.

Table 1.1 : Base Units of the SI System

Quantity	*Unit*	*Symbol*
Length	Metre	m
Mass	Kilogram	kg
Time	Second	s
Electric Current	Ampere	A
Temperature	Kelvin	K
Luminous Intensity	Candela	Cd
Amount of Substance	Mole	mol

The ***yard*** and ***mile*** as units of length are still in use in USA. These are given in Table 1.2.

Table 1.2 : Units of length still in use in daily life (USA).

1 mile	=	8 furlongs
1 furlong	=	220 yards
1 yard	=	3 feet
1 foot	=	12 inch
1 yard	=	0.9144 meter (exactly)
1 inch	=	2.54 cm (exactly)
1 mile	=	1.61 km

The guiding principle in choosing a unit of measurement is to relate it to common man's life as far as possible. As an example, take the unit of mass as ***kilogram*** or the unit of length as ***metre***. In our day to day business we buy food articles in kg or tens of kg. We buy cloth in metres or tens of metres. If gram had been chosen as the unit of mass or millimeter as unit of length, we would be unnecessarily using big numbers in our daily life. It is for this reason that the basic units of measurements are very closely related to our daily life.

The SI system is basically a metric system. The smaller and larger units of the basic units are multiples of ten only. They follow strictly the decimal system. These multiples or submultiples are given special names. These are listed in Table 1.3.

Table 1.3 : Prefixes for Powers of Ten

Power of ten	*Prefix*	*Abbreviation Symbol*	*Example*	
10^{-18}	atto	a		
10^{-15}	femto	f	femto metre	fm
10^{-12}	pico	*p*	pico farad	*p*F
10^{-9}	nano	n	nanometre	nm
10^{-6}	micro	μ	micron	μm
10^{-3}	milli	m	milligram	mg
10^{-2}	centi	c	centimetre	cm
10^{-1}	deci	d	decimetre	dm
10^{1}	deca	da	decagram	dag
10^{2}	hecto	h	hectometre	hm
10^{3}	kilo	k	kilogram	kg
10^{6}	mega	M	megawatt	MW
10^{9}	giga	G	giga hertz	GHz
10^{12}	tera	T	tera hertz	THz
10^{15}	peta	*P*		
10^{18}	exa	E		

1.3.2 Standards of Mass, Length and Time

Once we have chosen the fundamental units of the SI, we must decide on the set of standards for the fundamental quantities.

(i) Mass : The SI unit of mass is ***kilogram***. It is the mass of a particular cylinder made of Platinum - Iridium alloy, kept at the International Bureau of Weights and Measures in France.

This standard was established in 1887 and there has been no change because this is unusually stable alloy. Prototype kilograms have been made of this alloy and distributed to member states. The national prototype of India is the Kilogram no 57. This is preserved at the National Physical Laboratory, New Delhi.

(ii) Length : The metric system was established in France in 1792. The metre (also written as meter) was defined to be 1/107 times the distance from the Equator to the North Pole through Paris. This standard was abandoned for practical reasons. In 1872 an International Commission was set up in Paris to decide on more suitable metre standard. In 1875 the new metre was defined. It was defined as the distance between two lines on a Platinum-Iridium Bar stored under controlled condition. Such standards had to be kept under severe controlled conditions. Even then their safety against natural disasters is not gauranteed, and their accuracy is also limited for the present requirements of science and technology. In 1983 the metre was redefined as follows:

> ***One metre*** *is the distance travelled by light in vacuum in a time interval of 1/299792458 second. This definition establishes that the speed of light in vacuum is 299792458 metres per second.*

Following this definition a new prototype of ***one metre*** can always be prepared even if all the existing standards are destroyed in a natural disaster. This is the greatest advantage of this definition.

(iii) Time : The time interval ***second*** was originally defined in terms of the time of rotation of earth about its own axis. This time of rotation is divided in 24 parts, each part is called an ***hour***. An hour is divided into 60 minutes and each minute is subdivided into 60 seconds. Thus one ***second*** is equal to 1/86400 part of the solar day. To be more specific, the ***mean solar second***, the basic unit of time, was defined as $\frac{1}{60}\times\frac{1}{60}\times\frac{1}{24}$ part of the ***mean solar day*** for the year 1900. This definition was accepted upto 1960. It is known that the rotation of the earth varies substantially with time and therefore the length of a day is a variable quantity, may be very slowly varying.

The XIII General Conference on weights and measures in 1967 gave the following definition of the time interval '***second***'.

> ***One second*** *is the time required for Cesium - 133 atom to undergo 9192631770 vibrations.*

This definition has its roots in a device which can be named as ***atomic clock***. The frequency of certain atomic transitions can be measured with an accuracy of 1 part in 10^{12}. These frequencies (transition) are extremely stable and are least affected by the environment.

1.3.3 Derived Units

We have defined the three basic units in mechanics. When these basic units, interact, they give rise to quantities which are measured in derived units. Thus, the units which are obtained by the combination of the fundamental units, are called ***derived units.*** For example when distance and time interact, they give rise to speed acceleration etc. The speed is measured in metre per second (m/s). Similarly when length interacts with length, new quantities like area (m^2) and volume (m^3) etc result. The following tables give some of the derived units commonly used in mechanics and some derived units with special names.

Table 1.4 : Examples of derived SI Units

Quantity	*SI Unit*	*Symbol*
area	square meter	m^2
volume	cubic meter	m^3
speed or velocity	meter per second	m/s
acceleration	meter per square sec	m/s^2
density	kilogram per cubic meter	kg/m^3

Table 1.5 : Examples of derived SI Units with Special Names

Quantity	*Name*	*Symbol*	*Unit* Symbol
force	newton	N	kg m/s^2
pressure	pascal	Pa	N/m^2
energy, work	joule	J	Nm
Power	watt	W	J/s

The SI system of units form a coherent set in the sense that the product of any two unit quantities leads directly to the unit of the resulting quantity. When unit mass (kg) is divided by unit volume (m^3) we straight way get the unit of density kg/m^3. We should be careful in writing the units of certain quantities in proper order. Let us take the example of ***work***. The unit of work is Newton - meter which has been given a special name Joule. It should be written as Nm and not as mN. If written as mN it would mean "milli Newton".

Now, it is the time to check your progress. Solve the following intext questions and incase you have any problem, check answers given at the end of this lesson.

INTEXT QUESTIONS 1.1

1. A car is moving with a speed of 80 km/hr. What is the speed in m/s?

   ....

2. Distinguish between the fundamental and the derived units.

   ....

3. The radius of an atom is 10^{-10} m. What will be this value in terms of micro metre?

   ....

4. The total covered area of a house is 4500 square feet. Express this area in square metres.

   ....

1.4 DIMENSIONS OF PHYSICAL QUANTITIES

It is useful to assign dimension to physical quantities. The three basic dimensions of the three fundamental units, Length, Mass and Time are symbolized respectively as L, M, T. The dimensions of other physical quantities are expressed in terms of these symbols. See the following examples.

(1) $\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{m}{s} = \frac{L}{T} = LT^{-1}$

(2) $\text{density} = \frac{\text{mass}}{\text{volume}} = \frac{kg}{m^3} = \frac{M}{L^3} = ML^{-3}$

(3) $\text{force} = \text{mass} \times \text{acceleration} = kg \times \frac{m}{s^2} = MLT^{-2}$

The dimensional analysis is a very useful tool in checking the correctness of expressions or equations relating various physical quantities. Let us examine a few cases as examples.

Example 1.1: *The mechanical energy of a particle can be written in two different forms as (a) ½ mv² and (b) mgh. Are both dimensionally same?*

Solution :

(a) $\frac{1}{2} mv^2 = \frac{1}{2} M (L/T)^2 = \frac{1}{2} ML^2 T^{-2}$
(b) $mgh = (M)(L/T^2)(L) = ML^2 T^{-2}$

We therefore find that dimensionally both expressions for energy are equivalent. They differ only by a dimensionless multiplier, (Factor 1/2 in this case).

Example 1.2 : *Suppose a car starts from rest. The car covers a distance x in time t while moving with uniform acceleration a. Find an expression for x in terms of t and a.*

Solution : Suppose the expression for x is of the form,

$x \propto a^m t^n$, ($\propto$ is sign of proportionality)

This formula will be correct only if the dimensions on both sides are the same.

Left Hand Side (LHS) $x = L^1 = L^1 M^0 T^0$

Right Hand Side (RHS) $a^m t^n = (L/T^2)^m (T)^n$

$= L^m T^{-2m} T^n$

$= L^m T^{n-2m} = L^m M^0 T^{n-2m}$

If the two sides must have the same dimensions then by comparing the dimensions of L, M and T seprately we get

$m = 1, \quad n - 2m = 0$

$n - 2 = 0$

$n = 2$

Hence $x \propto at^2$.

We know (may come to know later) that this is not the proper form of the expression.

The correct expression is $x = \frac{1}{2} at^2$.

The sign of proportionality is replaced by the sign of equality with the help of a dimensionless multiplier (1/2 in this case).

Example 1.3 : *In an experiment with a simple pendulum we come to a qualitative conclusion that the time period **T** of the pendulum depends on the length* *of the pendulum l and the acceleration due to gravity g. But we do not know the exact dependance. Find the exact expression for T in terms of l and g*

Solution : Let us assume that

$T \propto l^m g^n$

Dimensionally L.H.S. $= T = L^0 M^0 T^1$

RHS $= l^m g^n = L^m (L/T^2)^n = L^{m+n} M^0 T^{-2n}$

By comparing the powers of L, M and T on both sides

$m + n = 0$ and $-2n = 1$

$m = -n$ $\quad n = -1/2$

$\therefore m = +1/2$ and $n = -1/2$

$\therefore T \alpha l^{1/2} g^{-1/2}$ or $T = 2k\sqrt{\frac{l}{g}}$

You should bear in mind that the numerical constant (2π) cannot be determined from dimensional analysis.

Example 1.4 : *It is known that a particle moving in a circular orbit has an acceleration which depends on the orbital speed v and radius r of the orbit. Determine the powers of v and r in the expression acceleration $a = kv^m r^n$ where k as usual is a dimensionless quantity.*

Solution :

LHS $\quad a = L/T^2 = LM^0 T^{-2}$

RHS $\quad v^m r^n = (L/T)^m L^n = L^{m+n} M^0 T^{-m}$.

Comparing the powers of L M and T on both sides, we get

$m + n = 1$ and $m = 2$

We get $n = -1$

$a = kv^2 r^{-1} = k\frac{v^2}{r}$

Again the numerical value of k cannot be obtained from dimensional analysis.

Now, take a pause and do the following questions.

INTEXT QUESTIONS 1.2

1. A stone is dropped from the roof of a tall building. The velocity v with which the stone hits the ground depends on the height h of the building and the acceleration due to gravity g. Obtain the expression for v.

   ....

2. The displacement of a moving particle is given by the expression $Y = A \sin(K_1 t + K_2 x)$ where Y, A and x are in meters and t in second. Obtain the dimensions of K_1 and K_2

   ....

3. The acceleration of a moving object is directly proportional to the applied force and inversely proportion to its mass. What is the dimension of force?

   ....

1.5 ORDER OF MAGNITUDE

It is quite often useful to know/calculate the approximate value of a particular quantity. It helps us in checking the results of any lengthy calculation or to know the approximate magnitude of a quantity. ***The order of magnitude*** of a quantity is the ***power of ten*** of the number that describes the quantity. We know that the speed of light in vacuum is 3×10^8 m/s. We, therefore, say that the order of magnitude is 8. If a quantity increases by ***four orders*** of magnitude we mean to convey that the quantity has increased by a factor 10^4.

Example 1.5 : *The size of an average room is $6 \times 5 \times 4 m^3$. The room is to be completely filled with cricket balls without crushing them. If the diameter of a ball is 5 cm, estimate the number of balls that will fill the room.*

Solution :

The volume of the room $= 6 \times 5 \times 4 = 120$ m^3

The approximate volume of a ball $= 5 \times 5 \times 5 \times 10^{-6}$ m^3

$= 125 \times 10^{-6}$ m^3

(You know that the actual volume of a ball is given by $\frac{4\pi r^3}{3}$ where r is the radius of the ball.)

Hence the total number of balls $= \frac{120}{125 \times 10^{-6}} \simeq 1 \times 10^6$ balls.

A more accurate calculation may give a result which may differ by factor of ten only.

Table 1.6 : Approximate values of certain measured quantities

Mean distance from earth to moon	3.8×10^8 m
Mean radius of the earth	6.4×10^6m
Diameter of a hydrogen atom	1×10^{-10} m
Diameter of an atomic nucleus	1×10^{-14} m
Mass of Sun	2×10^{30} kg
Mass of earth	6×10^{24} kg
Mass of a Mosquito	1×10^{-5} kg
Mass of an electron	9.1×10^{-31} kg
One year	3.2×10^7 s
One day	8.6×10^4 s
Period of a sound wave	1×10^{-3} s
Period of a radio wave	1×10^{-6} s

Now, check your learning by solving the following questions.

INTEXT QUESTIONS 1.3

1. A teacher consumes 0.4 unit of electricity per hour in his house. The cost of electricity is Rs 3/- per unit. Estimate his total expenditure on electricity per year.

   ....

2. Estimate the consumption of salt for the whole country in one year for a population of 900×10^6.

   ....

1.6 VECTORS AND SCALARS

1.6.1 Scalar Quantities

We know that the physical quantities are always described by some numbers with proper units attached with them. We describe the density of material, the volume of a jar, the distance between two cities, the speed of a moving car, the force that pulls all objects towards the earth and the torque that opens or closes a door. Some of these quantities are expressed in numbers with units and it is a complete and correct description of that quantity. The examples are (i) density of copper = 8.9×10^3 kg/m^3 (ii) mean radius of the earth = 6.4×10^6m (iii) one day = 8.6×10^4 s. Such quantities do not require any direction to be specified for their descritpion. These are known as ***scalar*** quantities.

> A ***scalar quantity*** has only magnitude and no direction.

1.6.2 Vector Quantities

A vector quantity is a physical quantity that is described by both the magnitude and the direction. ***Force*** is a vector quantity. If we apply a force of 100N on an object, we must also specify the ***direction*** in which the force is being applied. ***Velocity*** is an other vector quantity. If we describe the motion of an object we must specify how fast it is moving and the direction of its motion.

> A ***vector quantity*** has magnitude and direction both.

Note: A vector is written with an arrow above the letter to differentiate it with the [illegible]. Vector is written as A or A. Scalar is written as A. The magnitude of a vector is symbolised as |A| or |A|

Vectors can always be represented graphically. Let us say that a force of 500 N is applied on a body in the direction west to east. This vector quantity can be represented graphically as shown in Fig 1.1. below

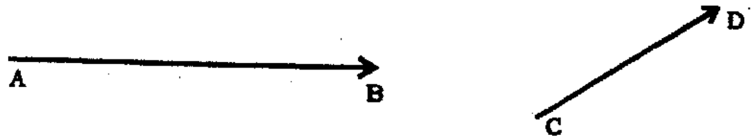


Fig 1.1: *Graphical representation of a vector.*

The line AB represents this vector. The length of the line AB, say 5 cm, represents the magnitude 500N. The direction of the vector is from A to B (West to east). The point A is called the ***tail*** and the point B with an arrow mark is called the ***tip*** (head) of the vector. Another vector CD of magnitude 300N force is represented by CD (3 cm length) pointing in a different direction.

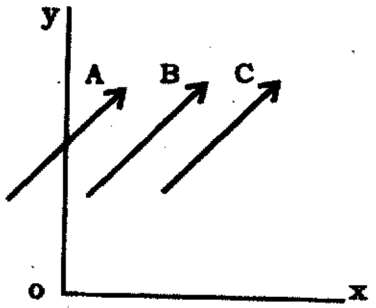


Fig. 1.2: *Equal vectors repre ented graphi ally.*

Any two vectors **A** and **B** are said to be **equal** if they have the same magnitude and point in the same direction. Graphically, therefore, all such vectors which are of the same length and parallel to each other are said to be equal as shown here in Fig 1.2. The three vectors **A**, **B** and **C** are of equal length (magnitude) and are parallel to each other (point in the same direction).

We therefore say that **A** = **B** = **C**.

1.6.3 Addition of Vectors

When two or more vectors are added together, like scalar quantities, they must have the same units. Graphically, it is very easy to add two vectors and find the resultant sum.

Let us have two vectors A and B and we have to find their vector sum **R = A + B**.

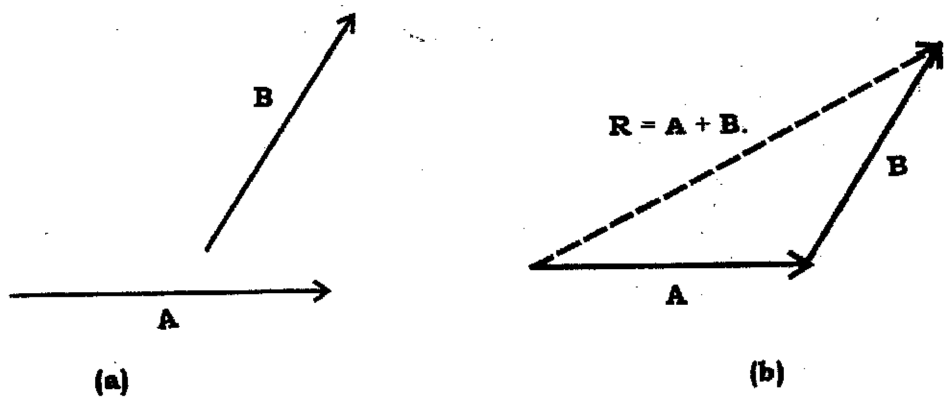


Fig 1.3 : *Addition of two vectors, graphically.*

The two vectors are shown in Fig. 1.3. (a). To find the sum of the two vectors we adopt the following procedure.

First, draw the vector **A** graphically. Then draw the vector **B** in such a manner that the *tail* of the vector **B** starts from the tip of Vector **A** as shown in fig. 1.3.(b). We know that a parallel movement does not change a vector. A vector drawn from the tail of **A** to the tip of **B** is the resultant vector **R** = **A** + **B**. Following the same procedure. We can find the sum of more than two vectors also.

Let us take three vectors **A**, **B** and **C** and we have to find the vector **R** = **A** + **B** + **C**

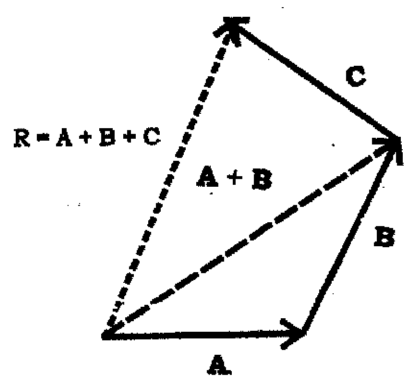


Fig 1.4 : *Addition of three vectors in a graphical manner*

Draw the vector **A**. Draw the vector **B** such that the tail of **B** starts from the tip of **A**. Then the vector which starts from the tail of **A** and ends at the tip of vector **B** is the sum (**A**+**B**). Following the same rule we can add the vector **C** to the Vector (**A** + **B**) and get to vector **R** = **A** + **B** + **C**. You can now appreciate that several vectors can be added together in the same manner.

Now, following the graphical procedure illustrated above, you can easily show that

(i) **A** + **B** = **B** + **A** (commutative of properly vector addition) (1.1)

(ii) **A** + (**B** + **C**) = (**A** + **B**) + **C** (Associative property of vector addition) (1.2)

1.6.4 Subtraction of Vectors

We define the ***negative of vector*** as a vector which has the same magnitude but points in the opposite direction. If **A** is a vector of magnitude 300 units pointing towards east, then –**A** is a vector of magnitude 300 units pointing towards west. Following this definition of a negative vector we can evaluate the difference of two vectors in the same way as we evaluated the sum of two vectors. Let us evaluate **R** = **A** – **B**

This can also be written as **R** = **A** + (–**B**)

The graphical construction of finding **A** – **B** is shown in Fig 1.5. First we draw the vector **A**. We then draw the Vector –**B**. To find **A** + (–**B**), we now draw the vector –**B** such that the tail of –**B** coincides with the tip of **A**. A vector which joins the tail of **A** to the tip of –**B** is the vector (**A** – **B**).

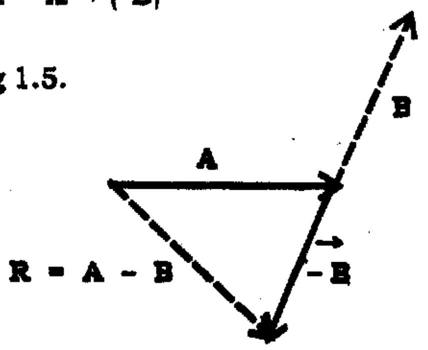


Fig 1.5 : *Subtraction of vectors graphically*

1.6.5 Law of Parallelogram of Vectors

We can find sum of two vectors anylitically without graphical construction also. Let us evaluate **R** = **A** + **B**

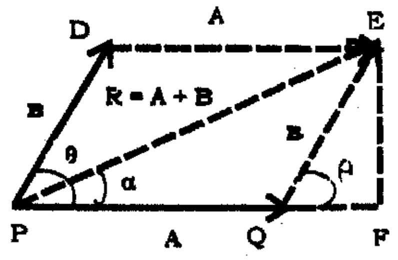


Fig 1.6 : *Law of parallelogram of vectors.*

Let **A** and **B** be two vectors and θ the angle between them. We construct a parallelogram PQED where **PQ** = **DE** = **A**

and **PD** = **QE** = **B**

Let EF be perpendicular to the line PQ. In ΔPFE

$$\begin{aligned}(PE)^2 &= (PF)^2 + (FE)^2 \quad (1.3)\\ &= (PQ + QF)^2 + (FE)^2\\ &= (PQ)^2 + (QF)^2 + 2(PQ)(QF) + (FE)^2\\ &= (PQ)^2 + [(QF)^2 + (FE)^2] + 2\,PQ.QF\\ &= (PQ)^2 + (QE)^2 + 2\,PQ.\,QF\\ &= (PQ)^2 + (QE)^2 + 2\,PQ.QE.(QF/QE)\\ R^2 &= A^2 + B^2 + 2AB\cos\theta \quad (1.4)\end{aligned}$$

$$|\mathbf{R}| = \sqrt{A^2+B^2+2AB\cos\theta} \quad (1.5)$$

This expression for R in equation (1.5) gives us the magnitude of the sum of two vectors **A** and **B** inclined at an angle θ between them.

Special Cases

i) When $\theta = 0$ the two vectors are parallel to each other. then $R = A + B$ (1.6 a)

ii) When $\theta = \pi/2$, $R = \sqrt{A^2+B^2}$ (1.6 b)

iii) When $\theta = \pi$ the two vectors are antiparallel to each other. They are directed opposite to each other.

$R = \sqrt{A^2+B^2-2AB}$, $R = A - B$ (1.6 c)

or $R = B - A$

The direction of the vector **R** with respect to a reference direction can also be determined analytically. Let the reference direction be the direction of one of the vectors, say vector **A** and **R** makes an angle α with **A**.

$$\tan\alpha = \frac{EF}{PF} = \frac{EF}{PQ+QF} = \frac{B\sin\theta}{A+B\cos\theta}$$

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta}$$

Since **A**, **B** and θ are known, the direction of **R** (angle α) can be easily calculated.

Special cases : We can discuss the same special cases for which we obtained the magnitude of R.

i) when $\theta = 0$, $\tan\alpha = 0$, $\alpha = 0$ (1.8a)
The direction of **R** coincides with **A**

ii) $\theta = \pi/2$, $\tan\alpha = B/A$ (1.8b)

iii) $\theta = \pi$, $\tan\alpha = 0$, **R** will be along the vector **A** or **B** (1.8c)

Again, it is time to check your understanding.

INTEXT QUESTIONS 1.4

1. There are three vectors of magnitudes 8, 16 and 20 units oriented arbitrarily in the same plane. What are the minimum and maximum values of their resultant?

....

2. A man walks 3.0 km towards east and then 4 km towards north.
   (a) How far away he is from the starting point?
   (b) What is the direction of his final position?

....

3. There are two vectors **A** and **B** of equal magnitude with an angle of 60° between them. Determine the following graphically
   (a) **A** + **B** (b) **A** - **B** (c) **B** - **A**
   (d) **A** + 2**B** (e) **A** - 2**B**

....

1.7 PRODUCT OF VECTORS

1.7.1 Scalar Product

The scalar product of two vectors **A** and **B** is defined as a scalar quantity which is a product of the magnitudes of the two vectors multiplied by the cosine of the angle between the two vectors

$$\mathbf{A}.\mathbf{B} = AB\cos\theta. \quad (1.9)$$

Where θ is the angle between the two vectors **A** and **B**. Fig 1.7.

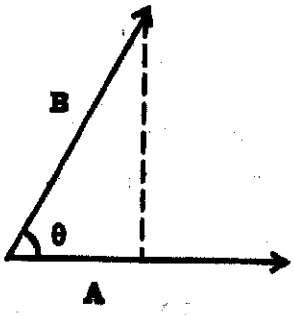


Fig 1.7 : *Scalar Product*

A dot symbol between **A** and **B** is put to indicate the scalar product. For this reason the ***scalar product*** is also called the ***dot product***. The quantities represented by the vectors **A** and **B** need not have the same units. The product AB cos θ contains the magnitudes of the vectors only and is a scalar quantity. It is therefore easy to appreciate the following relations.

$$\mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A} = AB\cos\theta \quad (1.10)$$

$$\mathbf{A}.(\mathbf{B}+\mathbf{C}) = \mathbf{A}.\mathbf{B} + \mathbf{A}.\mathbf{C} \quad (1.11)$$

We will find a practical use of these relations in later chapters on mechanics.

Example 1.6 : *The two vectors are of magnitudes* $|\mathbf{A}| = 30$ *and* $|\mathbf{B}| = 40$ *and the angle* $\theta = 55°$ *, calculate the magnitude and direction of the Vector* $\mathbf{R} = \mathbf{A} + \mathbf{B}$.

Solution :

As per equation (1.5) $|R| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$$= \sqrt{(30)^2 + (40)^2 + 2\times30\times40\times\cos 55°}$$

$$= (900 + 1600 + 2400 \times 0.5736)^{1/2}$$

$$|R| = 62.2 \text{ (magnitude)}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} = \frac{40\times0.8192}{30+40\times0.5736} = \frac{32.768}{52.944}$$

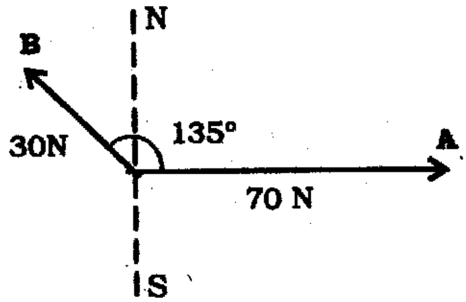
$= 0.6189, \quad \alpha = 31.8°$

The resultant **R** makes an angle of 31.8° with vector **A**.

Example 1.7 : *A cart is being pulled by man A towards east with a force of 70 N. The same cart is being pulled by man B towards North-west with a force of 30 N.*

(a) *Calculate the resultant force acting on the cart.*

(b) *Find the direction of this resultant force.*



Solution :

In this problem

$A = 70N, B = 30\ N, \ \theta = 135°$

$R = \sqrt{(70)^2 + (30)^2 + 2\times70\times30\cos135°}$

$= \sqrt{4900+900+4200\times\cos(90+45)°}$

$= \sqrt{5800-4200\sin45°} = \sqrt{2821} \qquad \therefore R = 53.1\ N.$

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} = \frac{30\times\sin(90+45)°}{70+30\cos(90+45)°} = \frac{30\cos45°}{70+30(-\sin45°)}$$

$$= \frac{30\times0.7071}{70-30\times.7071} = \frac{21.213}{48.787} = 0.4348$$

$\alpha = 23.5°$

1.7.2 Vector Product

The vector product of two vectors **A** and **B** is defined as a third vector **C** whose magnitude is AB sin θ where θ is the angle between the vectors **A** and **B**. The vector product is written as

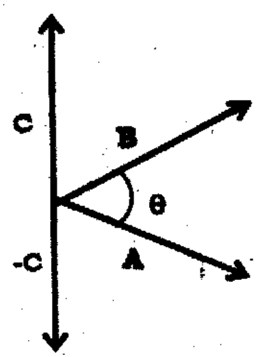


Fig. 1.8: Graphical representation of vector product.

$\mathbf{A}\times\mathbf{B} = \mathbf{C}$ (1.12)

The magnitude of the product is $|\mathbf{C}| = AB\sin\theta$ (1.13)

There is a simple rule to find the ***direction of the vector*** **C**. Stretch the fingers of your right hand along the vector **A**. Now curl the fingers from **A** towards **B** through the smaller angle θ between **A** and **B**. The *Thumb* which is erect points in the direction of the vector **C**. If one follows this rule one can easily see that

$\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ (1.14)

If **A** and **B** are in a horizontal plane with an acute angle θ between them as shown in the figure 1.8 then $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ will point vertically upwards and $\mathbf{B} \times \mathbf{A} = -\mathbf{C}$ will point vertically downwards. Because of a cross sign between **A** and **B**, the ***vector product*** is also called as ***cross product***. We will make use of these relations later.

1.8 RESOLUTION OF VECTORS

A vector can be resolved into components with respect to a particular co-ordinate system.

Fig 1.9 shows a vector **A** in a rectangular coordinate system. The tail of the vector coincides with the origin O of the co-ordinate system. Let us draw perpendiculars from the tip of the vector on the coordinate axes. We get quantities A_x and A_y. These are, called components of the vector **A**. The process is called ***resolution of vector into its components.***

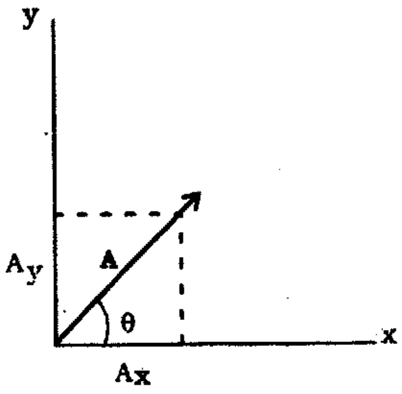


Fig 1.9 : *The components of the vector **A** along the rectangular coordinate axes x and y.*

The components A_x and A_y are

$$A_x = A\cos\theta$$

$$A_y = A\sin\theta \qquad (1.15)$$

where θ is the angle which the vector **A** makes with + X axis.

If we know the vector **A** i.e. we know the magnitude A and the direction with respect to a reference axis, we can obtain the components A_x and A_y. Conversely, if we know the components, we can obtain the magnitude and direction of the vector.

$$A_x = A\cos\theta$$

$$A_y = A\sin\theta$$

$$A_x^2 + A_y^2 = A^2(\cos^2\theta + \sin^2\theta) = A^2$$

$$A = \sqrt{A_x^2 + A_y^2} \text{ (magnitude)} \qquad (1.16)$$

→ $$\tan\theta = A_y/A_x \text{ (direction)} \qquad (1.17)$$

The sign of $\tan\theta$ determines the quadrant of the co-ordinate axis.

A more convenient way to handle the resolution of more than one vectors in a three dimensional space is to introduce the concept of unit vectors. The vector **A** can be written as

$$\mathbf{A} = \hat{u}_0|\mathbf{A}| \qquad (1.18)$$

Where $\hat{u}_0$ is a unit vector in the direction of **A**. (A cap over the letter represents a unit vector). In the rectangular co-ordinate system we choose $\hat{i}, \hat{j}$ and $\hat{k}$ as unit vectors along the x, y and z directions respectively. Let us do some simple mathematics for a two dimensional system. This can be easily extended to three dimensions.

Following the definition of unit vector.

$$\mathbf{A} = A_x\hat{i} + A_y\hat{j}$$

$$\mathbf{B} = B_x\hat{i} + B_y\hat{j} \qquad (1.19)$$

This is shown in the Fig 1.10.

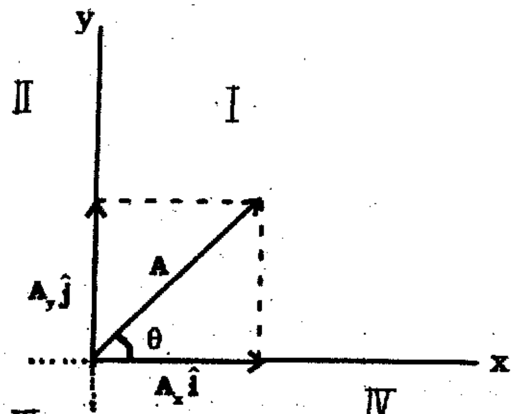


Fig. 1.10(a)

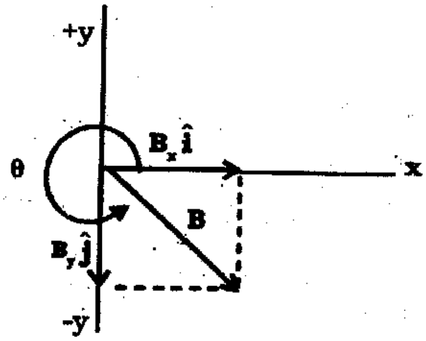


Fig. 1.10(b)

Figure 1.10 : Components of vectors along the rectangular co-ordinate axes. Vector A is in the quadrant I and the vector B in the quadrant IV.

Suppose we attempt to find the sum of the two vectors **A** and **B**.

$\mathbf{R} = \mathbf{A} + \mathbf{B}$
$R_x = A_x + B_x$
$R_y = A_y + B_y$

Let the vector **R** make an angle θ with X-axis.

$$|\mathbf{R}| = \sqrt{\left(R_x^2 + R_y^2\right)}$$

$$\tan\theta = \frac{R_y}{R_x}$$

We solve the following problem to illustrate the efficacy of this analytical method of addition of vectors.

Example 1.8: *Two vectors are given below*

$\mathbf{A} = 2\hat{i} + 3\hat{j}$ and $\mathbf{B} = 3\hat{i} - 4\hat{j}$

Find the sum of the vectors $\mathbf{A} + \mathbf{B} = \mathbf{R}$.

Solution :

According to our notation

$A_x = 2,\ A_y = 3;\ B_x = 3,\ B_y = -4$
$R_x = 2 + 3 = 5$
$R_y = 3 - 4 = -1$
$R = \sqrt{5^2 + 1^2} = \sqrt{26} = 5.1$
$\tan\theta = R_y/R_x = -1/+5 = -0.2$
θ = will lie in the IV quadrant.

Example 1.9 : *Two points A and B in the XY plane have co-ordinates (–3, 2)m and (2, 4) m respectively.*

(a) *Write expressions for position vectors of A and B ($\mathbf{R}_A$ & $\mathbf{R}_B$)*
(b) *Evaluate ($\mathbf{R}_A + \mathbf{R}_B$) and ($\mathbf{R}_A - \mathbf{R}_B$)*

Solution :

(a) The position vector $\mathbf{R}_A = (-3\hat{i} + 2\hat{j})$ m

$\mathbf{R}_B = (2\hat{i} + 4\hat{j})$ m

(b) $\mathbf{R}_A + \mathbf{R}_B = (-3\hat{i} + 2\hat{j}) + (2\hat{i} + 4\hat{j})$

$= (-2\hat{i} + 6\hat{j})$

$\mathbf{R}_A - \mathbf{R}_B = (-3\hat{i} + 2\hat{j}) - (2\hat{i} + 4\hat{j})$

$= -5\hat{i} - 2\hat{j}$.

Example 1.10 : *If a vector* **C** *is added to vector* **B** *the result is* $-9\hat{i} - 8\hat{j}$. *If* **B** *is subtracted from* **C** *the result is* $5\hat{i} + 4\hat{j}$. *What is the direction of* **B**.

Solution : Given,

$\mathbf{B} + \mathbf{C} = -9\hat{i} - 8\hat{j}$

$\mathbf{C} - \mathbf{B} = 5\hat{i} + 4\hat{j}$

By adding we set, $2\mathbf{C} = -4\hat{i} - 4\hat{j}$; $\mathbf{C} = -2\hat{i} - 2\hat{j}$

By subtracting we set, $2\mathbf{B} = -14\hat{i} - 12\hat{j}$; $\mathbf{B} = -7\hat{i} - 6\hat{j}$

Therefore, $\tan\theta = \dfrac{B_y}{B_x} = \dfrac{-7}{-6} = 1.167$

According to the tables given here θ is in the III quadrant.

$\theta = 221°$.

Example 1.11 : *Given*

(i) $\mathbf{A} + \mathbf{B} = 6\hat{i} + \hat{j}$

(ii) $\mathbf{A} - \mathbf{B} = -4\hat{i} + 7\hat{j}$

What is the (a) magnitude, and (b) direction of **A**

Solution :

(a) By additing (i) and (ii) we get $2\mathbf{A} = 2\hat{i} + 8\hat{j}$

$\therefore \mathbf{A} = \hat{i} + 4\hat{j}$

$A_x = 1$ and $A_y = 4$

$A = \sqrt{A_x^2 + A_y^2} = \sqrt{17}$, $A = 4..$

(b) $\tan\theta = \dfrac{A_y}{A_x} = \dfrac{4}{1}$, $\theta = \tan^{-1}(4)$.

INTEXT QUESTIONS 1.5

1. There is a wooden block on the floor. Person A is pulling it with a force **F** and person B is pushing it with the same force **F**. The direction of the force with respect to the floor is 30° in both cases. Who will be able to move the block on the floor more conveniently and why?

....

2. A vector **A** of magnitude 50 m is in the XY plane and makes an angle of 30° with the X - axis. What are the X and Y components of this vector?

....

3. A force of 200 N is applied on a body at an angle of 120° with the positive X - axis. What are the directions and magnitudes of the rectangular components of this force?

....

1.9 WHAT YOU HAVE LEARNT

- Every physical quantity is measured in some system of units. We in physics adopt the SI.
- In mechanics, kg, m and s are the base units of the SI for measuring mass, length and time respectively. The other units are called derived units
- A standard mass of an alloy material has been accepted as standard kilogram.
- One meter is the distance travelled by light in vacuum in $\frac{1}{299792458}$ second.
- One second is the time required for Cs –133 atom to undergo 9192631770 vibrations
- Every physical quantity has dimensions. The dimensional analysis is a useful tool to check the correctness of mathematical expressions.
- There are vector and scalar quantities.
- Addition and subtraction of vectors — Law of parallelogram of vectors.
  (i) Scalar product $\mathbf{R} = \mathbf{A}.\mathbf{B} = AB \cos\theta$.
  (ii) Vector product $\mathbf{R} = \mathbf{A} \times \mathbf{B}$
  $|\mathbf{R}| = AB \sin\theta$. The direction of **R** is also defined.
- A vector can be resolved into its components.

1.10 TERMINAL QUESTIONS

1. The average speed of the fastest train in India is 120 km/h. What will be the speed in metre per second?

2. Show that the expression $s = ut + \frac{1}{2} at^2$, where s is distance, u is speed and a is the acceleration of a moving particle, is dimensionally correct.

3. If $x = At + Bt^3$ (where x is distance and t is time) is dimensionally correct, then show that $(A + 3Bt^2)$ has the dimensions of velocity.

4. The diameter of an atom is about 10^{-10} m. Estimate the number of atoms in 1 m^3 of a solid.

5. A big boat is resting on still water. There are two ropes tied at the same point on the boat. One man pulls the boat along south with a force 12 N and another man pulls it along east with a force of 16 N.
   (a) What is the resultant force with which the boat is being pulled?
   (b) What is the direction of its displacement ?

6. A river is flowing towards east with a speed of 5 km/h . A swimmer is swimming from the south bank with a speed 4 km/h always heading towards north.
   (a) Will he be able to reach the north bank exactly at the opposite point?
   (b) What is the direction of his effective speed in water?
   (c) What is his effective speed in the river?

7. A boy is pulling a table towards south with a force 5 N and his sister is pulling the same table toward, east with a force 4N on the horizontal floor of the house. There is a force of friction of 1N between the floor and the table in every direction. What is the magnitude of the resultant force which is moving the table?

8. A man walking across his fields in a dark night loses his way. He walks 100 steps towards west and then 60 steps towards north and then some steps in another direction so that he again comes back to the starting point. How many steps and in

what direction he walked in his third movement? Solve the problem graphically.

9. A person walks along the paths shown in Fig 1.11, starting from the point A and ending at the point B. Find the magnitude of his displacement.

10. A particle moves with a velocity v along east for 5 seconds and then towards north with velocity v for 5 seconds.
   (a) How much is the change in velocity per second.
   (b) Change in velocity per second is defined as acceleration. What is the direction of the acceleration?

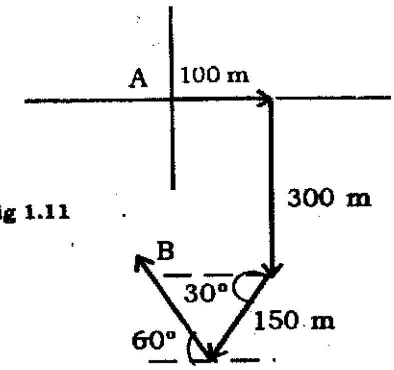


Fig 1.11

11. A block of mass 5 kg is sliding on an inclined plane making an angle of 30° with horizontal. What is the magnitude of the force driving the block along the plane?

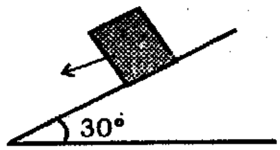


Fig 1.12

12. Two vectors are **A** = 8 units and **B** = 5 units. The angle between them is 60°. Find
   (a) **A.B**
   (b) **A** × **B** and its direction, for the following orientation (**A** and **B** are in horizontal plane).

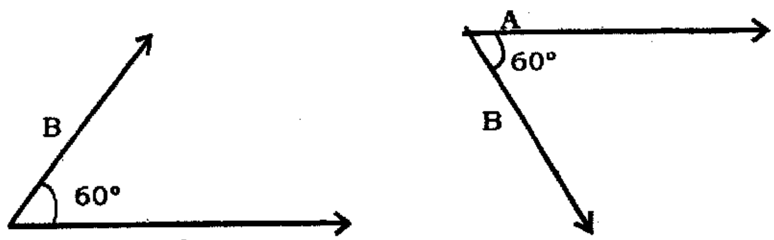


Fig 1.13

13. Two vectors are given by $\mathbf{A} = 3\hat{i} - 2\hat{j}$ and $\mathbf{B} = 2\hat{i} - 5\hat{j}$. Calculate (**A** + **B**) and (**A** − **B**)

ANSWERS TO THE INTEXT QUESTIONS

Intext Questions 1.1

1. $\frac{80\,km}{h} = \frac{80 \times 1000}{60 \times 60} = \frac{200}{9} m/s$
   = 22.2 m/s
2. The fundamental units are independent units and they are only seven in number. Whereas the derived units are obtained by the combination of the fundamental units and there can be any number of derived units.
3. 10^{-10}m = 1 micro metre.
4. 1 feet × 1 feet = 1 sq. feet
   $= \frac{1 \times 12 \times 12 \times 2.54 \times 2.54}{10^4}$
   $= \frac{929}{10^4} = .0$[illegible] ^{2}

4500 sq feet = 45× 9.29 = **418 m²**

Intext Question 1.2

1. $v \propto g^m h^n$

   $LHS = \frac{L}{T} = M^0 L^1 T^{-1}$

   $RHS = g^m h^n = \left(\frac{L}{T^2}\right)^m . L^n$

   $= L^{m+n} T^{-2m}$

   Compare $M^0 L^1 T^{-1} = L^{m+n} T^{-2m}$
   (LHS) (RHS)

   $m + n = 1$, $-2m = -1$
   $n = 1 - m$, $m = ½$
   $n = ½$

   $\therefore v \propto g^{½} h^{½}$ or $v^2 \propto gh$

2. $Y = A \sin (K_1 t + K_2 x)$

since displacement is also in meter.
LHS = $Y = L^1 M^0 T^0$
RHS = $A \sin (K_1 t + K_2 x) = L^1$
Quantity in ( ) must be dimensionless.
$K_1 t = L^0 M^0 T^0$, Dimension of $K_1 = T^{-1}$
$K_1 x = M^0 L T^0$, Dimension of $K_2 = L^{-1}$

3. acceleration $\propto \dfrac{\text{Applied force}}{m}$

$$a \propto \frac{F}{m}$$

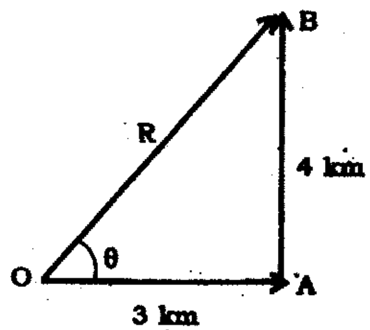
Force $= ma = M\,L\,T^{-2}$

Intext Questions 1.3

1. 0.4 units per hour.
   In one day $0.4 \times 24 = 9.6$ units
   $\simeq$ 10 units/day.
   Per year 3650 units.
   Cost 3650×3 = Rs 10950 = Rs 10^4.
2. A person may consume about 1 kg/months, 10 kg/year. For a population of 900×10^6
   Total salt = 9000×10^6 Kg
   $\simeq 10^{10}$ kg.

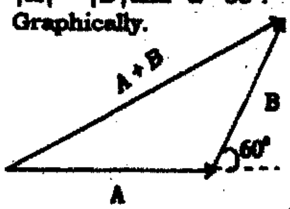
Intext Questions 1.4

1. 8, 16, 20 magnitude.
   Maximum of result will be when all three are parallel = 44 units.
   Minimum when antiparallel = 4 units.

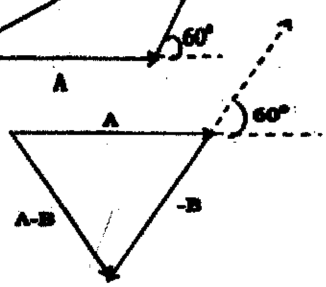


2. (a) R = 5 km.
   (b) $\tan\theta = 4/3 = 1.33$
   θ = about 53°
3. |A| = |B| and α = 60°.
   Graphically.

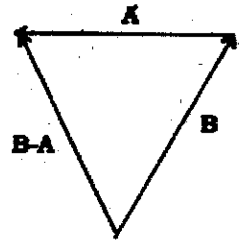
(a)



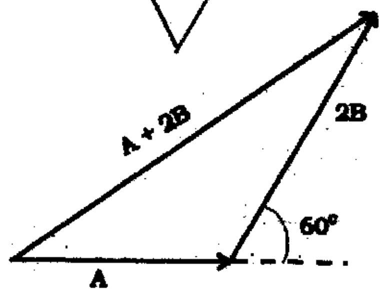
(b)



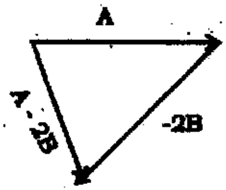
(c)



(d)



(e)



Intext Questions 1.5

1.

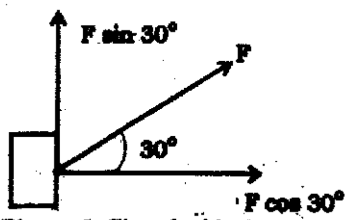


Fig a : Pulling the block

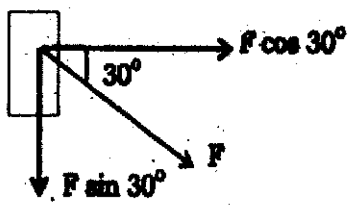
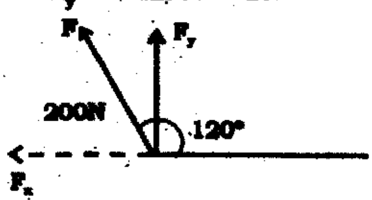


Fig b : Pushing the block

From the above figures it is clear that pulling the block is convenient than pushing it on the floor.

2. $A_x = 50 \cos 30° = 25\sqrt{3}$
   $A_y = 50 \sin 30° = 25$.
3.



$F_x = 200 \cos 120° = -100$
$F_y = 200 \sin 120° = 100\sqrt{3}$.