

22

HARMONIC MOTION

22.1 INTRODUCTION

In your first book of Physics you have studied about different types of motion like motion in a straight line, projectile motion and circular motion which are mainly determined by the path followed. But there are some motions which repeat after a certain interval of time. For example the motion of the hands of a clock, to and fro motion of the swing and pendulum's bob.

In this lesson you will study about the periodic motions particularly the oscillatory motions with examples from your daily life. You will also study about the simple harmonic motion and related concepts. The next lesson on elastic waves will deal with different types of mechanical waves, their characteristics and uses.

22.2 OBJECTIVES

After studying this lesson, you should be able to,

- explain periodic motion, with examples;
- show that an oscillatory motion is necessarily periodic but a periodic motion may not be necessarily oscillatory;
- define simple harmonic motion and show simple harmonic motion as projection of uniform circular motion on a diameter of a circle;
- explain the basic concepts of simple harmonic motion;
- derive expressions of time periods of harmonic oscillators;
- derive expressions for the potential and kinetic energy of a simple harmonic oscillator;
- explain damped harmonic oscillations.

22.3 PERIODIC MOTION

You would have observed a pendulum clock and noticed that the pointed end of its second hand and that of its minute hand move around in a circle each with a fixed period. The second hand completes its journey around the dial in one minute but the minute hand takes one hour to complete one round trip. However, the pendulum's bob moves to and fro or back and forth and completes its motion from one end to the other and back to its first end in a fixed time. *Such a motion or a phenomenon which repeats itself after a fixed interval of time is called **periodic motion**.* There are two classes of periodic motion : (i) non-oscillatory (ii) oscillatory. The motions of the hands of the clock are circular and **non-oscillatory** but the to and fro motion of the pendulum's bob is **oscillatory**. However, both the motions are periodic. It is important to note that an oscillatory motion is necessarily periodical but a periodic motion is not necessarily oscillatory. However, we must hasten to add that a periodic motion can be shown as a combination of a number of oscillatory motions. Remember that a motion which repeats itself in equal intervals of time is **periodic** and if it is also to and fro over the same path it is **oscillatory**.

At this stage of learning you are aware of the fact that the Earth completes its rotation about its own axis in 24 hours. Thus, alternate days and nights are formed. It also revolves around the Sun and completes its revolution in 365 days. This second motion of the Sun produces a sequence of seasons. Similarly all the planets move around the Sun in elliptical orbits and each completes its revolution in a fixed interval of time.

Harmonic Motion: However to and fro motion of a pendulum is oscillatory and also periodic. We shall see that the displacement from the equilibrium position of an oscillatory motion can be represented by expressions containing sines and cosines of angles. Such expressions in mathematics are known as **harmonic functions**. Therefore, a particle executing a motion of this type is known as **harmonic oscillator** and its to and fro motion is called **harmonic motion** or **harmonic oscillation**. We shall first consider simple harmonic motion (abbreviated as SHM). But before discussing this type of motion, we shall perform an activity.

Activity: Suppose that the displacement y of a particle, executing simple harmonic motion, from its mean position can be represented by the equation :

$$y = a \sin \theta \quad (22.1)$$

or
$$y = a \cos \theta \quad (22.2)$$

From your book of mathematics find the values of $\sin \theta$ and $\cos \theta$ for $\theta = 0, 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 180^\circ, 240^\circ, 270^\circ, 300^\circ, 330^\circ$ and 360° . Then assuming that $a = 2.5$ cm, determine the values of y corresponding to each angle using the relation $y = a \sin \theta$ choosing a suitable scale plot graph between y and θ . Similarly using the relation $y = a \cos \theta$ plot another graph between y and θ . Notice carefully that each of the two graphs represents an oscillation between $+a$ and $-a$. Conclusion is inescapable that a certain type of oscillatory motion can be represented by an expression containing sine or cosine of an angle or by a combination of such expressions.

Now take a break and check your progress by solving the following questions.

INTEXT QUESTIONS 22.1

1. What is difference between a periodic motion and an oscillatory motion?
.....
2. Which of the following examples represent a periodic motion?
(i) A bullet fired from a gun, (ii) An electron revolving around the nucleus in an atom, (iii) A vehicle moving with a uniform velocity on a road, (iv) A comet moving around the Sun, (v) Motion of an oscillating mercury column in a u-tube.
.....
3. Is an oscillatory motion periodic or non-periodic?
.....
4. What is a harmonic oscillator?
.....

22.4 SIMPLE HARMONIC MOTION

It has been stated that the oscillations of an harmonic oscillator can be represented by the terms containing sines and cosines of angles. If the displacement of an oscillatory particle from its mean position can be represented by an equation $y = a \sin \theta$ or $y = a \cos \theta$ or $y = A \sin \theta + B \cos \theta$, where a , A and B are constants, then the particle executes *simple harmonic motion*. We define simple harmonic motion (abbreviated as SHM) as under :

A particle is said to execute simple harmonic motion if it moves to and fro about a fixed point under the action of a force (or torque) directly proportional to the displacement of the particle from the fixed point and the direction of force is opposite to that of displacement.. We shall restrict our discussion to linear oscillations.

To derive the equation of simple harmonic motion, let us consider a point M moving with a constant speed v in a circle of radius a (Fig. 22.1), and centre O . At $t = 0$, let the point be at x and at $t = t$ the point be at M . The position vector OM specifies the position of the moving point at the time $t = t$. It is obvious that the position vector OM , called the **phasor** rotates with a constant angular velocity $\omega = v/a$. (This representation is also used in analysis of ac circuits). The acceleration of the point M is $v^2/a = \omega^2 a$ towards the centre O . At the time $t = t$,

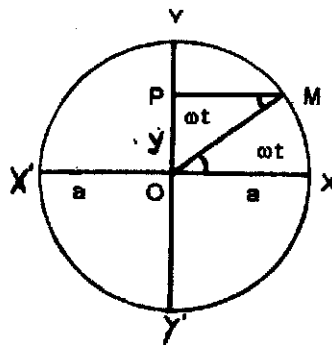


Fig. 22.1 : Simple harmonic motion of P along YOY'

the component of this acceleration towards $OY = a\omega^2 \sin \omega t$. Let us draw MP perpendicular to YOY . Then P can be regarded as a particle of mass m moving with an acceleration $a\omega^2 \sin \omega t$. The force on the particle P is therefore,

$$F = ma\omega^2 \sin \omega t, \text{ towards } O$$

But, $\sin \omega t = y/a$,

$$\text{therefore, } F = ma\omega^2 y/a = m\omega^2 y \quad (22.3)$$

The displacement is measured from O towards, P

Therefore,

$$F = -m\omega^2 y \quad (\text{see the negative sign})$$

Since, this force is directed toward O , and proportional to the displacement ' y ' of P from O , hence, the particle P is executing simple harmonic motion.

Let us put $m\omega^2 = k$, a constant, then from eq. (22.3) we get,

$$F = -ky \quad (22.4)$$

The constant k which is force per unit displacement is called the **force constant**

As $m\omega^2 = k$, therefore (22.5)

$$\omega^2 = k/m$$

In one complete rotation, OM describes an angle 2π and it takes a time T to complete one rotation. Hence

$$\omega = 2\pi/T \quad (22.6)$$

But $\omega = \sqrt{k/m}$, therefore,

$$T = 2\pi\sqrt{m/k} \quad (22.7)$$

This is the time taken by P to move from O to Y , then through O to Y' and back to O .

22.4.1 Basic Concepts of a Simple Harmonic Motion

Let us now define the basic concepts of a simple harmonic motion. These are the following:

Displacement : It is the distance, linear or angular at a given instant of the harmonic oscillator from its mean or equilibrium position.

Amplitude : It is the maximum displacement of the oscillator from the mean position on either side of the mean position.

Time period : It is the time taken to complete one vibration (oscillation) It is denoted by T .

Frequency : It is the number of vibrations (oscillations) completed by the oscillator in one second. It is denoted by ν . Its SI unit is hertz

(symbol Hz). Since ν is the number of vibrations per second, therefore, the time taken to complete one vibration is $1/\nu$. Hence $T = 1/\nu$ or $\nu = 1/T \text{ s}^{-1}$. As harmonic oscillations can be represented by expressions containing $\sin \theta$ and / or $\cos \theta$, as such we need two more important concepts. These are the following :

Phase Angle ϕ : It is the angle whose sine or cosine at a given instant indicates the position and direction of motion of the oscillator. It is expressed in radians.

Angular Frequency ω : It is the rate of change of phase angle. It is expressed in radian per second. Since in one complete vibration the phase angle ϕ changes from 0 to 2π radians, therefore, the rate of change of phase angle is $\omega = 2\pi / T = 2\pi \nu$ or $\omega = 2\pi \nu$.

Example 22.1: A tray of mass 9 kg is supported by a spring of force constant k as shown in Fig. 22.2. On pressing the tray slightly downward and then releasing, it executes S.H.M. of period 1.0s. When a block of mass M is placed on the tray the period increases to 2.0 s. Calculate the mass of the block.

Solution : The angular frequency of the system is $\omega = \sqrt{k/m}$, where m is the mass of the oscillatory system. $\omega = \sqrt{k/m}$

$\omega = 2\pi / T$, therefore, from eq. (22.7) we get

$$4\pi^2 / T^2 = k/m \text{ or } m = \frac{kT^2}{4\pi^2}$$

When the tray is empty $m = 9 \text{ kg}$ and $T = 1 \text{ s}$

$$\text{Therefore, } 9 = \frac{k \cdot 1^2}{4\pi^2}$$

On placing the block, $m = 9 + M$ and $T = 2 \text{ s}$.

$$\text{Therefore, } 9 + M = k \times (2)^2 / 4\pi^2$$

From the above two equations we get

$$\frac{(9+M)}{9} = 4$$

Therefore, $M = 27 \text{ kg}$.

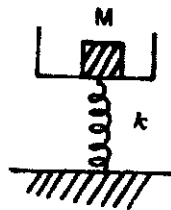


Fig. 22.2

Example 22.2: A spring of force constant 1600 Nm^{-1} is mounted on horizontal table as shown (Fig. 22.3) A mass $m = 4.0 \text{ kg}$ attached to the free end of the spring is pulled horizontally towards the right, through a distance of 4.0 cm and then set free. Calculate (i) the frequency (ii) maximum acceleration and (iii) maximum speed of the mass.

Solution: $\omega = \sqrt{k/m} = \sqrt{1600/4} = 20 \text{ rad s}^{-1}$;

Therefore $\nu = 20/2\pi = 3.18 \text{ Hz}$; $f_{\text{max}} = a\omega^2 = .04 \times 400 = 16 \text{ ms}^{-2}$, $v_{\text{max}} = a\omega = .04 \times 20 = 0.8 \text{ ms}^{-1}$.



22.4.2 Simple Harmonic Motion as Projection of Uniform Circular Motion on a Diameter of a Circle

Let us prove this fact using trigonometry. Let us consider a point M (Fig. 22.4) moving anti-clockwise with constant speed ω in a circle of radius a . Let us suppose that at $t = 0$, the point P coincides with x and at $t = t$, the point is at a position shown in the figure. Then $MOX = \omega t$. But OM is a vector which specifies at the given time $t = t$, the position of the moving point M relative to the centre O . From M drop MP perpendicular to YOY' . As the point M moves, the foot P of the perpendicular MP too moves to and fro along YOY' . At any given instant of time the displacement y of P from O is the component of the vector OM . The rotating vector OM whose component represents the actual motion of P along YOY' is called the **phasor**. (A quantity which varies sinusoidally with time may be represented as the projection of a rotating vector called phasor. This representation you will also use in the analysis of a.c. circuits). The circle around which the point moves is called the **reference circle**.

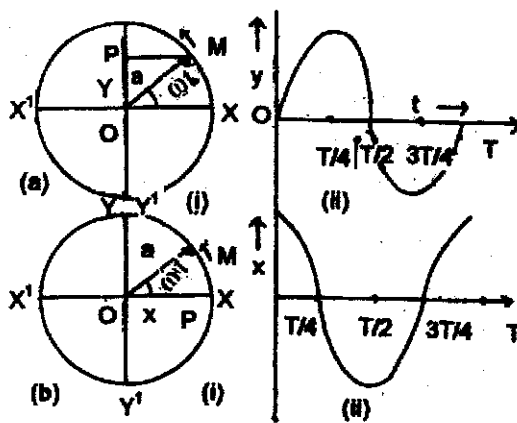


Fig. 22.4: SHM as projection of circular motion a (i), b (ii) and displacement from graph a (ii), b (ii)

Let us refer to Fig. 22.4(a). In this position of the rotating vector, or the phasor, $MOX = \omega t$ and the displacement of P from O is $OP = y$. From POM , we find that

$$y = a \sin \omega t \quad (22.8)$$

In Fig. 22.4 (b), in time t the phasor OM too describes an angle ωt but the displacement x of P from O is

$$x = a \cos \omega t \quad (22.9)$$

Eq. (22.8) represents simple harmonic motion of P along YOY' and equation (22.9) simple harmonic motion of P along XOX' .

Fig. 22.4 a(ii) and Fig. 22.4 b (ii) represent the corresponding graphs between the displacement y and x versus the time t . Now let us suppose that at $t = 0$, M is at M_0 such that $y = y_0$. Then from Fig. 22.5 (a) we get

$$y = a \sin (\omega t + \phi) \quad (22.10)$$

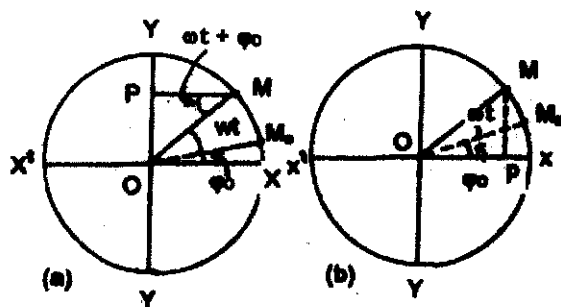


Fig. 22.5: SHM as the projection of uniform circular motion on the diameter.

and from Fig. 22.5

$$x = a \cos (\omega t + \phi) \quad (22.11)$$

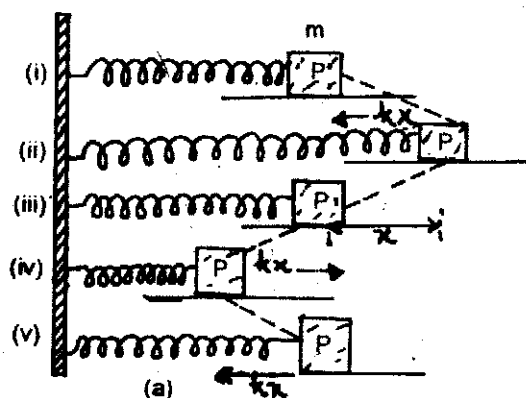
From the foregoing discussion we find that simple harmonic motion can be regarded as projection of a uniform circular motion on a diameter of circle. Such a circle is called the *reference circle*.

22.5 EXAMPLE OF SHM

In order to clarify the concept of SHM, some very common examples are given below.

22.5.1 Horizontal Oscillations of a Mass Point Attached to a Spring

Consider a horizontal elastic spring of spring constant k attached to a small block P of mass m . The other end of the spring is attached to a rigid wall. (Fig. 22.6a). The mass of the spring is negligible in comparison with the mass of the block. The block can be made to move on a smooth horizontal plane.



22.6 (a) : Oscillations of a point mass attached to a spring

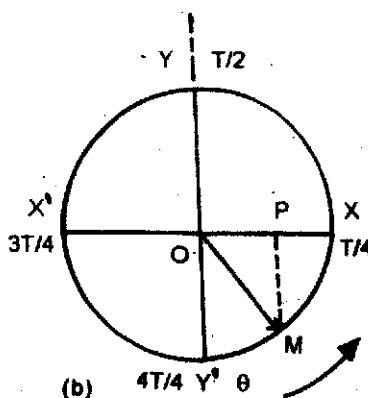


Fig. 22.6(b): oscillations of point mass P as projection of uniform circular motion on the diameter XOX'

Let us suppose that the spring is massless and there occurs no loss of energy against air resistance and friction. Initially, that is, at $t = 0$, the block is at rest (Fig. 22.6 a(i)) and then pulled horizontally through a small distance a to the right (Hooke's Law) [Fig. 22.6 a(ii)]. The work done against the elastic force changes into the potential energy of the block-spring system. As the spring has undergone an extension a , it exerts on the block a force kx towards left tending to restore the block to its equilibrium position. As the block returns to its initial position [Fig. 22.6 a(iii)] it acquires a velocity v and hence kinetic energy $K = (1/2) mv^2$ equals to loss in potential energy. Owing to inertia of motion it continues moving towards left till it arrives at the position shown in (Fig. 22.6 (a) iv). In this position, the block again experiences a force kx which brings it back to the initial position [Fig. 22.6 (a) v]. And, thereafter, the block continues oscillating to and fro. The time period of oscillation is $2\pi\sqrt{m/k}$, where k is the force

per unit extension of the spring. It is called the *force constant of the spring*.

22.5.2 Simple Pendulum

It consists of a point mass suspended by an *inextensible, massless* string from a *fixed* point about which it can oscillate *freely*. Such an ideal pendulum is difficult to construct. However, a small spherical bob suspended by a long cotton thread held between the two halves of a clamped split cork in an stand is approximately a simple pendulum (Fig. 22.7).

When the bob is displaced through a *small distance* from its equilibrium position and then set free, the pendulum executes angular oscillations in a vertical plane about its equilibrium position. Time for about 20 oscillations is measured using a stop watch reading upto one tenths of a second. Thus the time period T , the time for one complete oscillation is determined. The distance between the point of suspension and the centre of gravity of the bob is measured. It is l , the length of the pendulum. The forces acting on the bob of the pendulum in the displaced position shown in Fig. 22.7 are :

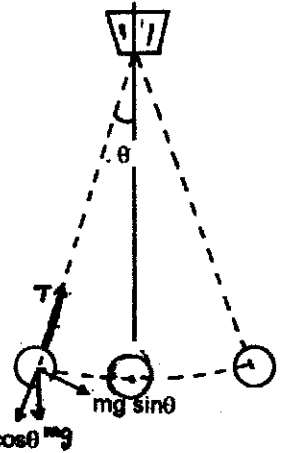


Fig. 22.7: Simple Pendulum

- (i) mg the weight of the bob vertically downwards
- (ii) T is the tension of the string upwards along the string. Note that the weight mg and equal and opposite reaction by the cork in the equilibrium position constitute a tension. In the displaced position $mg \cos \theta$ and equal and opposite reaction of the bob constitute a tension.

The weight mg is resolved into components : (a) $mg \cos \theta$ along the string and (b) $mg \sin \theta$ perpendicular to the string as shown. The component $mg \cos \theta$ balances the tension, T and the component $mg \sin \theta$ produces acceleration in the bob in the direction of the arrow. For *small* displacement x of the bob, the restoring force is $mg \theta = mg x/l$. The force per unit displacement is $k = mg/l$ and hence

$$\omega = \frac{\sqrt{k}}{m} = \sqrt{\frac{mg/l}{m}} = \sqrt{\frac{g}{l}} \text{ or } \frac{2\pi}{T} = \frac{\sqrt{g}}{l}$$

$$\text{Hence, } T = 2\pi \sqrt{\frac{l}{g}} \quad \dots(22.12)$$

22.5.3 Vertical Oscillation of a Mass Suspended from a Spring

Let us suspend a spring of force constant k from a rigid support [Fig. 22.8(a)]. Then let us attach a block of mass m to the free end

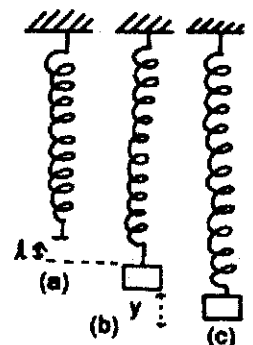


Fig. 22.8: Vertical oscillation of a block suspended from the free end of a spring.

of the spring. As a result of this the spring undergoes an extension say l [Fig. 22.8 (a)]. Obviously the force constant of the spring is $k = mg/l$. Let us now pull down the block through a small vertical distance y (Fig. 22.8 (c)). On the block, now acts a force ky vertically upwards. Therefore, on releasing the block, the force ky pulls it upwards. As the block returns to its initial position it continues moving upwards on account of the velocity it has gained. It overshoots the equilibrium position almost by y . The gravity again pulls it downwards and again the block overshoots the equilibrium position by almost the same vertical distance y . Thus, the vertical oscillations of the block continue. The angular frequency of this vertical oscillation is

$$\omega = \sqrt{k/m} = \sqrt{g/l}$$

Hence
$$T = 2\pi \sqrt{\frac{l}{g}} \quad (22.13)$$

We have assumed that the mass of the spring is negligible in comparison with the mass of the block.

22.5.4 Vertical Oscillations of a Liquid Column in a U-Tube

Consider a liquid column in a U-tube (Fig. 22.9). The liquid column in the left arm is pushed down through a small vertical distance y by a piston. Since the liquid is incompressible it rises up in the other arm through the same distance. Consequently, the difference of level of the liquid in the two arms of the U-tube is $2y$. If a be the area of cross-section of the U-tube, and ρ be the density of the liquid, then the force on the liquid on the right is $2y \rho ga$ vertically downwards. Now, if the piston is taken out from the left arm, the liquid column begins to oscillate. Let us find the period of oscillation of the liquid column. If l be the length of the liquid column, and ρ the density of the liquid then the mass of the liquid column is $m = \rho al$. As ω^2 is the return force per unit mass per unit displacement, therefore,

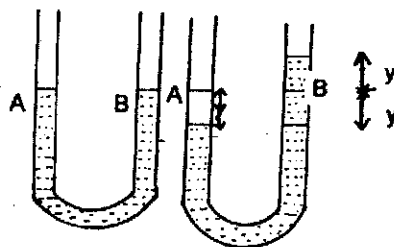


Fig. 22.9: Vertical oscillations of a liquid column in a U-tube.

$$\omega^2 = \frac{2y\rho ga}{\rho al}, \text{ or } \omega = \sqrt{\frac{2g}{l}} \text{ and hence the time period of oscillation is}$$

$$T = 2\pi \sqrt{l/2g} \quad (22.14)$$

Example 22.3: The figure 22.10, shows an oscillatory system comprising of two blocks of masses m_1 and m_2 joined by a massless spring of spring constant k . The blocks are pulled apart each with a force F and then released. What is the angular frequency of each mass. The blocks are on a smooth horizontal plane.

Solution: Let x_1 and x_2 be the displacement of each block when pulled

apart. The extension produced in the spring in $x_1 + x_2$. Thus the acceleration of m_1 is $k(x_1 + x_2)/m_1$ and the acceleration of m_2 is $k(x_1 + x_2)/m_2$. Since the same spring provides the restoring force to each mass, hence the net acceleration of the system comprising of the two masses and the massless spring equals to the sum of the accelerations produced in the two masses. Thus the acceleration of the system is $k(x_1 + x_2) [1/m_1 + 1/m_2] = kx/\mu$ where $x = x_1 + x_2$ is the extension of the spring and μ is the reduced mass of the system given by $1/\mu = 1/m_1 + 1/m_2$. The angular frequency of each mass of the system is therefore

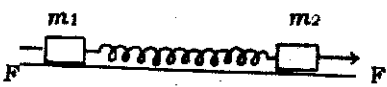


Fig. 22.10: Oscillatory system of masses attached to a spring.

$$\omega = k/\mu \quad \dots(22.15)$$

Now, take a pause and check your progress

INTEXT QUESTIONS 22.2

1. A small spherical ball of mass m is placed on a smooth spherical bowl of radius r a little away from the bottom point. Find the time period of oscillation of the ball. (Fig. 22.11)

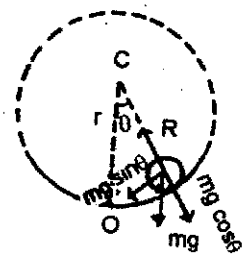


Fig. 22.11:

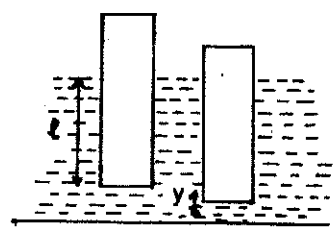


Fig. 22.12:



Fig. 22.13:

2. A cylinder of mass m floats vertically in a liquid of density ρ . The length of the cylinder inside the liquid is L . Find its time period of oscillation. (Fig. 22.12)
3. Find the frequency of oscillation of the mass m connected to two rubber bands as shown in the figure. The force constant of each of the two bands is k . (Fig. 22.13)

22.6 ENERGY OF SIMPLE HARMONIC OSCILLATOR

As you have seen, one form of the equation of simple harmonic motion is

$$y = a \sin \omega t \quad (22.16)$$

when t changes to $t + \Delta t$, Δy changes to $y + \Delta y$. Therefore

$$y + \Delta y = a \sin \omega (t + \Delta t) = a \sin (\omega t + \omega \Delta t)$$

$$= a [\sin \omega t \cos \omega \Delta t + \cos \omega t \sin \omega \Delta t]$$

As $\Delta t \rightarrow 0$, $\cos \omega \Delta t \rightarrow 1$ and $\sin \omega \Delta t \rightarrow \omega \Delta t$. Then

$$y + \Delta y = a \sin \omega t + a \cos \omega t \cdot \omega \Delta t \quad (22.17)$$

Subtracting eq. 22.16 from eq. 22.17 we get

$$\Delta y = \Delta t \omega a \cos \omega t$$

$$\text{or } \Delta y / \Delta t = \omega a \cos \omega t$$

$$\text{or } v = \omega a \cos \omega t \quad (22.18)$$

where $v = \Delta y / \Delta t$ is the velocity of the oscillator at time $t = t$. Hence, the **kinetic energy** of the oscillator at that instant of time is,

$$K = 1/2 m v^2 = (1/2) \omega^2 a^2 \cos^2 \omega t \quad (22.19)$$

Let us now calculate the potential energy of the oscillator at that time. When the displacement is y , the restoring force is ky , where k is the force constant. For this purpose we shall plot a graph of restoring force ky versus the displacement y . We get a straight line graph shown in Fig. 22.14. On the straight line graph OC, let us take two points P and Q and drop PM and QN perpendicular to Oy. As the points P and Q are close to each other, therefore, PQNM can be regarded as a rectangle. The area of this rectangular strip is $ky \cdot \Delta y$. This area equals to the work done against the restoring force ky when the displacement changes by a small amount Δy . The area of the triangle OBC is, therefore, equal to the work done in the time the displacement changes

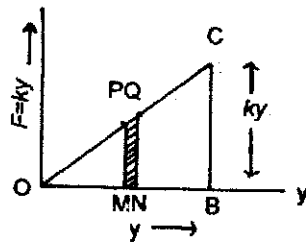


Fig. 22.14: Graph between the displacement y and the restoring force ky

from O to OB ($=y$) $= \frac{1}{2} ky^2$. This work done against the conservative force is the **potential energy** U of the oscillator. Thus, the potential energy of the oscillator when the displacement is y is,

$$U = \frac{1}{2} ky^2 \quad (22.20)$$

But $\omega^2 = k/m$, therefore, substituting $k = m\omega^2$ in the eq.(22.20) we get

$$U = \frac{1}{2} m\omega^2 y^2$$

Further as $y = a \sin \omega t$, therefore

$$U = \frac{1}{2} m\omega^2 a^2 \sin^2 \omega t$$

As the kinetic energy when the displacement is y from eq. 22.19 is

$$K = \frac{1}{2} m\omega^2 a^2 \cos^2 \omega t$$

Therefore, the total energy of the oscillator at any instant is

$$E = U + K = \frac{1}{2} m\omega^2 a^2 (\sin^2 \omega t + \cos^2 \omega t) = \frac{1}{2} m\omega^2 a^2 \quad (22.21)$$

The graph of kinetic energy K , potential energy U and the total energy E versus the displacement y is shown in Fig. 22.15. From the graph it is evident that for $y = 0$, $K = E$ and $U = 0$. As the displacement y from the mean position increases the kinetic energy decreases but potential energy increases. At the mean position the potential energy is zero but the kinetic energy is maximum. At the extreme positions, the energy is wholly potential. However the sum $K + U = E$ is constant.

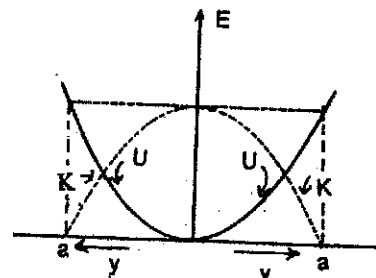


Fig. 22.15: The variation of potential energy U and kinetic energy K with displacement from equilibrium position

INTEXT QUESTIONS 22.3

1. When is the kinetic energy of a harmonic oscillator maximum; in its equilibrium position or when the displacement is maximum? When is the acceleration maximum?
2. Why does the amplitude of a simple pendulum decrease with time? What happens to the energy of the pendulum when its amplitude decreases?

22.7 DAMPED HARMONIC OSCILLATIONS

Take a simple harmonic oscillator comprising of a metal block B suspended from a fixed support S by a spring G . [Fig. 22.16(a)]. Place below the block a tall glass cylinder filled two thirds with water so that the block is about 6 cm below the surface of water and about the same distance above the bottom of the beaker. Paste a millimetre scale on the side of the cylinder just opposite to the pointer attached to the block. Push the block a few centimetres downwards and then release it. After

each oscillation note down the uppermost position of the pointer on the millimetre scale and the time. Then plot a graph between time ' t ' and the amplitude a of oscillation. The graph [Fig. 22.16(b)] shows that the amplitude decreases with time. Such oscillations are called **damped oscillations**.

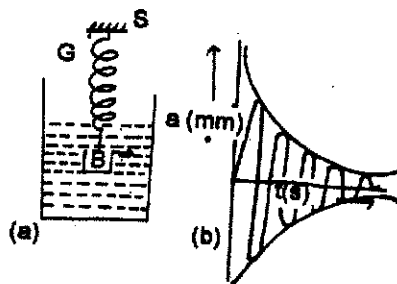


Fig. 22.16: Damped vibrations
(a) experimental setup
(b) damped oscillations

22.7.1 Free, Forced Vibrations and Resonance

To understand the difference between these phenomena, let us perform the following activity.

From a flexible support (Fig. 22.17) suspend four pendulums A, B, C and D. The pendulums A and B are of equal lengths, C has a shorter and D has a longer length. The pendulum B has a heavy bob. Set pendulum B into oscillations. You will observe that after a few minutes the other three pendulums too begin to oscillate. However, the amplitude of A is the largest. Why? Each pendulum is an oscillatory system with natural frequency of its own. The pendulum B which has a large mass transmits its vibrations on each of the

pendulum A, C and D. As a consequence of this, the pendulums C and D are forced to oscillate *not with their respective natural frequency but with the frequency of the pendulum B*. The phenomenon is called **forced oscillation**.

By holding the bob of any one of these pendulum you can force it to oscillate with any frequency you desire. This too

is a case of forced oscillations. The frequency of the force impressed by the pendulum B on the pendulums C and D is different from the respective natural frequency of C or of D. Both C and D are forced to oscillate with the frequency of B. However, the pendulum A on which too the oscillations of the pendulums B are impressed, oscillates with a relatively large amplitude with **its natural frequency**. This phenomenon is known as **resonance**.

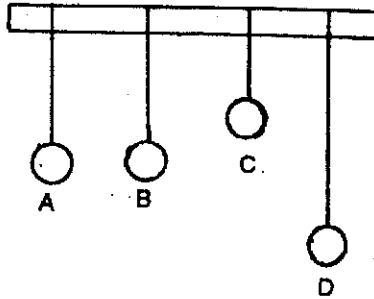


Fig. 22.17: Experimental arrangement for forced vibrations and resonance.

To sum up when the moving part of an oscillatory system is displaced from its equilibrium position and then set free, it oscillates to and fro about its equilibrium position with a frequency that depends upon certain parameters of the system only and not upon any external oscillating force. Such oscillations are known as **free vibrations**. The frequency with which the system oscillates is known as **natural frequency of the system**. The oscillatory system on which the oscillations are impressed is called **driven** and the system which applies the oscillating force is known as the **driver**. The particular case in which the natural frequency of the driver, equals the natural frequency of the driven, is known as **resonance**.

INTEXT QUESTIONS 22.4

1. Distinguish between resonance and forced vibration.

2. When the stem of a vibrating tuning fork is pressed against the top of a table a loud sound is heard. Does this observation demonstrate

the phenomenon of resonance or forced vibrations? Give reasons for your answer. What is the cause of the loud sound produced?

.....

3. Why are certain musical instruments provided with sound boards or sound boxes?
-

22.8 WHAT YOU HAVE LEARNT

- Periodic motion is a motion which repeats after equal intervals of time.
- Oscillatory motion is to and fro motion on the same path. An oscillatory motion is essentially periodic but a periodic motion is not necessarily oscillatory.
- Harmonic motion is an oscillatory motion in which the displacement of the oscillator from its mean position can be represented by expressions containing sines and cosine of angles.
- Simple harmonic motion is to and fro motion under the action of a restoring force or torque proportional to the displacement of the particle from its equilibrium position and directed towards the mean position.
- Time period is the time taken by the given particle to complete one oscillation.
- Frequency is the number of vibrations completed by the oscillator in 1s.
- Phase angle is the angle whose sine or cosine at the given instant indicates the position and direction of motion of the particle.
- Angular frequency is the rate of change of phase angle. Note that $\omega = 2\pi/T = 2\pi\nu$, where ω is the angular frequency in rads^{-1} , ν is the cyclic frequency in hertz (symbol : Hz) and T is the time period in seconds.
- Simple harmonic motion can be regarded as projection of uniform circular motion on a diameter of the circle.
- Equation of simple harmonic motion is

$$y = a \sin (\omega t + \phi_0) \text{ or } y = a \cos (\omega t + \phi_0)$$

Where y is the displacement from the mean position at a time t , ϕ_0 is the initial phase angle (at $t = 0$). A simple harmonic motion has only a single frequency of vibration and constant amplitude. In general, the vibrations produced may consist of a mode of lowest frequency ν called the fundamental or first harmonic plus higher harmonics of frequencies $2\nu, 3\nu, 4\nu, \dots$

- When an oscillatory system vibrates of its own, without the influence of an external oscillatory system, its vibrations are said to be free. If, however, an oscillatory system A is driven by an outside system B called the driver and the system A is forced to

vibrate with the frequency of B, the vibrations of A are said to be forced vibrations. However, if the frequency of the driver equals to the natural frequency of the driven, the phenomenon is known as *resonance*.

An Oscillator, such as a pendulum, while oscillating has to overcome the viscous resistance of air. As a consequence of this, energy is lost as heat. Hence the amplitude of vibration decrease with time. Such vibrations whose amplitude decreases owing to loss of energy as heat in overcoming frictional and viscous forces are called damped vibrations.

22.9 TERMINAL QUESTIONS

- Distinguish between a periodic and oscillatory motion.
- What is a simple harmonic motion?
- Which of the following functions represent (i) simple harmonic motion (ii) periodic but not simple harmonic (iii) non periodic motion? Give the period of each periodic motion.
 (1) $\sin \omega t + \cos \omega t$ (2) $1 + \omega^2 + \omega t$ (3) $3 \cos \left(\omega t - \frac{\pi}{4} \right)$
- The time period of oscillation of mass 0.1 kg suspended from a Hooke's law spring is 1s what would be time period of oscillation of mass 0.9 kg when suspended from the same spring?
- What is phase angle? How is it related to angular frequency?
- Why is the time period of a simple pendulum independent of the mass of the bob, when the period of a simple harmonic oscillator is $T = 2\pi \sqrt{m/k}$?
- When is the magnitude of acceleration of a particle executing simple harmonic motion maximum? When is the restoring force maximum?
- Show that simple harmonic motion is the projection of a uniform circular motion on a diameter of the circle. Obtain an expression for the time period of a simple harmonic oscillator in terms of mass and force constant.
- Obtain expressions for the instantaneous kinetic energy potential energy and the total energy of a simple harmonic oscillator.
- Show graphically how the potential energy U , the kinetic energy K and the energy E of a simple harmonic oscillator vary with the displacement from equilibrium position.
- The displacement of a moving particle from a fixed point at any instant is given by $x = a \cos \omega t + b \sin \omega t$. Is the motion of the particle simple harmonic? If your answer is no, explain why? If your answer is yes, calculate the amplitude of vibration and the phase angle
- In an oscillating simple pendulum, the amplitude is 0.04 m and

the time period is 10 s. Calculate the maximum velocity.

13. Imagine a ball dropped in a frictionless tunnel cut across the Earth through its centre. Obtain an expression for its time period in terms of radius r of the Earth and the acceleration due to gravity ' g ' on the surface of the Earth.
14. A ball of mass m can move freely in the neck of area of cross-section a of an air chamber of volume V . When pressed down a little and then released the ball executes simple harmonic motion. Assuming that changes in pressure and volume are isothermal, calculate the period of oscillation of the ball.
15. In the figure 22.18 is shown a block of mass $m = 2$ kg connected to two springs each of force constant $k = 400$ Nm⁻¹. The block is displaced by 0.05 m from equilibrium position and then released. Calculate (a) The angular frequency ω of the block, (b) its maximum speed; (c) its maximum acceleration; and total energy dissipated against damping when it comes to rest.



Fig. 22.18

16. In Fig (22.19) (a) and (b) are shown two different arrangements of two springs joined to a block of mass m . If the block in each arrangement is displaced a little downwards and then released, what would be the angular frequency of oscillation in each case?

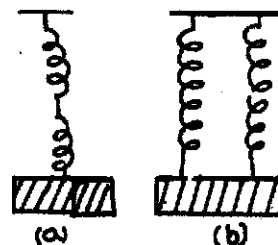


Fig. 22.19

CHECK YOUR ANSWERS

INTEXT QUESTIONS 22.1

1. A motion which repeats after fixed interval of time is a periodic motion. A to and fro motion on the same path is an oscillatory motion.
2. (ii), (iv) (v); 3. Periodic ; 4. A particle whose motion can be represented by expressions containing sines and or cosines of angles.

INTEXT QUESTIONS 22.2

1. Return force on the ball when displaced a distance x from the equilibrium position is $mg \sin \theta = mg \theta = mg x/r$.
2. On being pushed down through a distance y , the cylinder experiences an upthrust $y a \rho g$. Therefore $\omega^2 = y a \rho g / y m$. From the law of flotation $m = a l r$. Hence, $\omega^2 = g/l$ or $T = 2\pi\sqrt{l/g}$.
3. $\omega^2 = k/m$ and hence $\nu = 1/2\pi\sqrt{k/m}$. Note that when the mass is displaced, only one of the bands exerts the restoring force).

INTEXT QUESTIONS 22.3

- .. K.E is maximum at mean position or equilibrium position; acceleration is maximum when displacement is maximum.
2. As the pendulum oscillates it does work against the viscous resistance of air and friction at the support from which it is suspended. This work done is dissipated as heat. As a consequence of this the amplitude decreases.

INTEXT QUESTIONS 22.4

1. When an oscillatory system called the driver applies its oscillations of force on another oscillatory system called the driven and the second system is forced to oscillate with the frequency of the first system, the phenomenon is known as forced vibrations. In the particular case of the forced vibrations in which the frequency of the driver equals to the frequency of the driven system, the phenomenon is known as resonance.
2. The table top is forced to vibrate not with its natural frequency but with the frequency of the tuning fork. Therefore, this observation demonstrates forced vibrations. Since a large area is set into vibrations, hence the intensity of the sound increases.
3. The sound board or box is forced to vibrate with the frequency of the note produced by the instrument. Since a large area is set into vibrations, the intensity of the note produced increases and its duration decreases.

TERMINAL QUESTIONS

3. (i) periodic and SHM, $\frac{2\pi}{\omega}$; (ii) non periodic; (iii) periodic and SHM,

$$T = \frac{2\pi}{\omega}$$

4. 3.0 s

11. Put $a = A \sin \phi_0$ and $b = A \cos \phi_0$. Then $x = A \sin (\omega t + \phi)$ and

$$\phi_0 = \tan^{-1} \frac{a}{b}, \text{ and } A = \sqrt{a^2 + b^2}$$

12. $v_{\max} = a\omega = 0.025 \text{ ms}^{-1}$

13. [Hints At a depth d below the Earth's surface $g_d = g(1-d/r) =$

$$\frac{g(r-d)}{r} = g/r \times x \text{ where } x \text{ is the distance from the centre of the}$$

Earth. Hence $\omega^2 = g/r$ and $T = 2\pi\sqrt{r/g}$

14. For a displacement y , change in volume is $\Delta V = y\alpha$. The restoring force is then $F = \Delta p \cdot a = E \frac{\Delta V}{V} \alpha = E \frac{\alpha^2 y}{V} \therefore \omega^2 = \frac{F}{my} = \frac{E\alpha^2}{mV}$ and hence,

$$T = 2\pi \sqrt{\frac{mV}{E\alpha^2}}$$

15. (a) $\omega = 20 \text{ rads}^{-1}$, (b) 1.0 ms^{-1} , (c) 20.0 ms^{-2} (d) 1.0 J .

16. $\sqrt{k_1 k_2 / m(k_1 + k_2)}$; $\sqrt{(k_1 + k_2) / m}$.