4

MOTION IN A PLANE

4.1 INTRODUCTION

In the previous two lessons you have studied concepts related to motion in a straight line and Newton's Laws of Motion. Certainly these are not enough to describe motion in a plane. Can you describe the motion of objects moving in a plane using the concepts discussed in these lessons? How ever, you have also learnt to represent motion graphically in two dimensions. So we will carry forward from there and introduce certain new concepts in this lesson to describe motion in a plane, i.e., motion in two di mensions. In this lesson, you will mainly study about projectile motion and circular motion. We will introduce the related concepts of angular speed, centripetal acceleration, centripetal and centrifugal force to explain this kind of motion.

Studying this lesson will help you answer interesting questions like the following : What should the position and speed of an aircraft be so that the food or medicine packets dropped from it reach people affected by floods or earthquakes? How should an athlete throw a discuss or a javelin so that it covers the maximum horizontal distance? How should roads be designed so that cars taking a turn around a curve do not go off the road? What should the speed of a satellite be so that it moves in a circular orbit around the earth? And many other such questions. In the next lesson you will study about another law of nature discovered by Isaac Newton, nam the universal law of gravitation.

4.2 OBJECTIVES

After studying this lesson, you should be able to,

- *define projectile motion and circular motion and give examples of both;*
- *derive expressions for the time of flight, range and maximum height of a projectile;*
- *derive the equation of the trajectory of a projectile;*
- *derive expressions for velocity, and acceleration of a particle in circular motion;*
- *define radial and tangential acceleration.*

4.3 PROJECTILE MOTION

The motion of a cricket ball hit for a six, a javelin thrown by an athlete, a food packet dropped by an aeroplane are some common examples of projectile motion. To study their motion, we have to first define projectile motion. So let us begin by asking : What is a projectile motion? You know that near the surface of the earth, every object falls freely with a constant downward acceleration of about 9.81 m s^{-2} due to gravity. Now suppose you throw a ball from some initial position with some initial velocity. Then if all other effects, namely, effects of wind and friction due to air are ignored, the acceleration of the ball is equal to g, the acceleration due to gravity.

Thus, the motion of the ball is ***motion with constant vertical acceleration and zero horizontal acceleration.*** This kind of motion is called ***projectile motion. In general, we define projectile motion as motion which has constant velocity in a certain direction and constant acceleration in the direction perpendicular to that of velocity.***

A football hit by a child, a speeding bullet, experience only constant vertical acceleration with constant horizontal velocity. Hence they are all examples of projectile motion (see Fig 4.1). Can you think of some more examples?

In most such cases, the body moves with an initial vertical component of velocity. But we can also launch the body (also called **the projectile**) horizontally without any initial upward component of velocity.

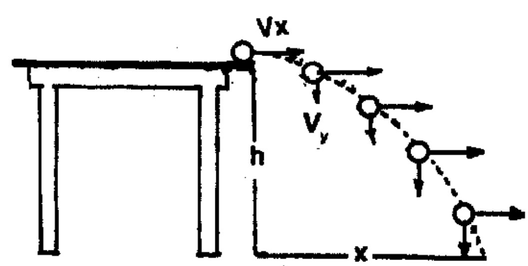

Fig 4.1: *Some examples of projectile motion.*

Having defined projectile motion, we would like to determine how high and how far it goes and how long it remains in the air. These factors are important if we want to launch a projectile to land at a certain target - for instance, a football in the goal!

4.3.1 Maximum Height, Time of Flight and Range of a Projectile

Let us analyse projectile motion to determine its maximum height, time of flight and range. In doing so, we will be ignoring all other effects such as wind or air resistance. We can characterise the initial velocity of an object in projectile motion by its vertical and horizontal components. Let us take the positive x-axis in the horizontal direction and the positive y-axis in the vertical direction (Fig 4.2).

Let us assume that the initial position of the projectile is at the origin 0 at $t = 0$. As you know, the coordinates of the origin are $x = 0$, $y = 0$. Now suppose the projectile is launched with an initial velocity v_0 at an angle θ_0, known as the ***angle of elevation***, to the x-axis. Its components in the x and y direction are,

$$v_{0x} = v_0 \cos \theta_0 \quad (4.1\ a)$$
$$v_{0y} = v_0 \sin \theta_0 \quad (4.1\ b)$$

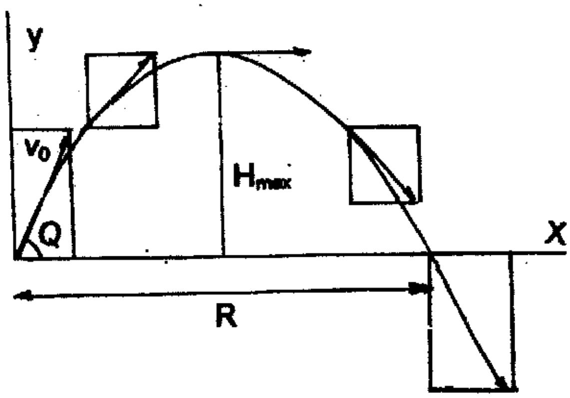

Fig 4.2 : *Maximum height, time of flight and range of a projectile.*

Let a_x and a_y be the horizontal and vertical components, respectively, of the projectile's acceleration. Then

$$a_x = 0,\ a_y = -g = -9.81\ ms^{-2} \quad (4.2)$$

The negative sign for a_y appears as the acceleration due to gravity is in the negative y direction in the chosen coordinate system.

Notice that a_y is constant. Therefore, we can use Eqs (2.6 and 2.9) to write expressions for the horizontal and vertical components of the projectile's velocity and position at time t. These are given by

Horizontal motion $\quad v_x = v_{0x},\quad$ since $a_x = 0 \quad (4.3\ a)$

$$x = v_{0x} t \quad (4.3\ b)$$

Vertical motion $\quad v_y = v_{0y} - gt \quad (4.3\ c)$

$$y = v_{0y} t - \tfrac{1}{2} g t^2 \quad (4.3\ d)$$

The vertical position and velocity components are also related through Eq (2.10) as

$$-gy = \tfrac{1}{2} (v^2_y - v^2_{oy}) \quad (4.3\ e)$$

You will note that the horizontal motion (given by Eqs. 4.3 a and b) is motion with constant velocity. And the vertical motion (given by Eqs. 4.3 c and d) is motion with constant (downward) acceleration. The vector sum of the two respective components would give us the velocity and position of the projectile at any instant of time.

Now let us make use of these equations to find out the maximum height, time of flight and range of a projectile.

(a) Maximum height : As the projectile travels through the air, it climbs upto some maximum height (h) and then begins to come down. ***At the instant when the projectile is at the maximum height, the vertical component of its velocity is zero.*** This is the instant when the projectile stops to move upward and does not yet begin to move downward. Thus, putting $v_y = 0$ in Eqs. (4.3 c and d), we get

$$0 = v_{0y} - gt$$

or $$t = \frac{v_{0y}}{g} = \frac{v\sin\theta_0}{g} \quad (4.4)$$

Thus, the maximum height of the projectile is

Maximum height $$h = \frac{v_0^2 \sin^2\theta_0}{2g} \quad (4.5)$$

Note that in our calculation we have ignored the effects of air resistance. This is a good approximation for a projectile with a fairly low velocity.

Using Eq. (4.4) we can also determine the time for which the projectile is in the air. This is termed the ***time of flight***.

(b) Time of flight : ***The time of flight of a projectile is the time interval between the instant of its launch and the instant when it hits the ground.*** The time t given by Eq. (4.4) is just the time for half the flight of the ball. Therefore, the total time of flight is

Time of flight $$T = 2t = \frac{2v_0 \sin\theta_0}{g} \quad (4.6)$$

Finally we have to find out the distance travelled horizontally by the projectile. This is also called its ***range***.

(c) Range : The range R of a projectile is calculated simply by multiplying its time of flight by its horizontal velocity. Thus,

$$R = (v_x)\ (2t)$$

$$= (v_0\cos\theta_0)\frac{(2v_0\sin\theta_0)}{g}$$

$$= v_0^2\frac{(2\sin\theta_0\cos\theta_0)}{g}$$

Since, $2 \sin\theta \cos\theta = \sin 2\theta$, the range is

Range $$R = \frac{v_0^2 \sin 2\theta_0}{g} \quad (4.7)$$

You can see that the range of a projectile depends on

- *its initial speed* v_0, and
- *its direction given by* θ_0

Now can you determine the angle at which a disc or a javelin should be thrown so that it covers maximum distance horizontally? In other words, for what angle would the range be maximum?

Clearly, R will be maximum for any speed when $\sin 2\theta_0 = 1$ or $2\theta_0 = 90°$. Thus for *R to be maximum at a given speed* v_0, $\theta_0 = 45°$.
Let us determine these quantities for a particular case.

Example 4.1: *In the centennial Olympics held at Atlanta in 1996, the gold medallist hammer thrower threw the hammer to a distance of 19.6 m. Assuming this to be the maximum distance, calculate the initial speed with which the hammer was thrown. What was the maximum height of the hammer? How long did it remain in the air? Ignore the height of the thrower's hand above the ground.*

Solution: We can ignore the height of the thrower's hand above the ground. Thus the launch point and the point of impact are at the same height. We take the origin of the coordinate axes at the launch point. Since the distance covered by the hammer is maximum, it is equal to the hammer's range for $\theta_0 = 45°$. Thus we have from Eq (4.7).

$$R = \frac{v_0^2}{g} \qquad \text{or} \qquad v_0^2 = Rg, v_0 = \sqrt{Rg}$$

It is given that $R = 19.6$ m. Putting $g = 9.8$ ms^{-2} we get

$$v_0 = \sqrt{(19.6)\text{m}\ (9.8\ \text{ms}^{-2})} = 14.01\text{ms}^{-1}$$

The maximum height and time of flight are given by Eqs. (4.5) and (4.6), respectively. Putting the value of v_0 and $\sin\theta_0$ in Eqs (4.5) and (4.6) we get

$$h = \frac{(190.08)\text{m}^2\,\text{s}^{-2} \times \left(\frac{1}{2}\right)^2}{2 \times 9.8\ \text{ms}^{-2}} = 2.45\ \text{m}$$

$$T = \frac{2 \times \left(9.8\sqrt{2}\right)\text{ms}^{-1}}{9.8\text{ms}^{-2}} \times \tfrac{1}{2} = 1.43\,\text{s}$$

Now, that you have studied some concepts related to projectile motion and their application, you may like to check your understanding. Try solving the following problems.

INTEXT QUESTIONS 4.1

1. Identify examples of projectile motion from among the following situations.
   (a) An archer shoots an arrow at a target
   (b) Rocks are ejected from an exploding volcano
   (c) A truck moves on a mountainous road
   (d) A bomb is released from a bomber plane
   (e) A boat sails in a river

   ....

2. Which of the following statements about projectile motion are true?
   (i) The horizontal component of velocity changes with time.
   (ii) The vertical component of velocity changes with time.
   (iii) The range depends only on the angle of elevation.
   (iv) The time of flight depends only on initial velocity.
   (v) The time of flight depends only on the vertical component of initial velocity.

   ....

3. Three balls thrown at different angles reach the same maximum height (Fig. 4.5).
   (a) Are the vertical components of the initial velocity the same for all the balls? If not, which one has the least vertical velocity component?
   (b) Will they all have the same time of flight?
   (c) Which one has the greatest horizontal velocity component?

   ....

1. **In the 1968 Olympics in Mexico City, Bob Beamon broke the record for the long jump with a jump of 8.90 m. Assume his initial speed on take off to be 9.5 ms^{-1}. How close did he come to the maximum possible range in the absence of air resistance? The value of g in Mexico city is 9.78 ms^{-2}.**

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Although we have discussed quite a few things about projectile motion, we have still not answered the original question: What is the path or trajectory of the projectile? So let us determine the equation for a projectile's trajectory.

4.3.2 The Trajectory of a Projectile

Do you recognise the shapes of the projectile trajectories of Figs 4.1, 4.2 and 4.3?

It is easy to determine the equation for the path or trajectory of a projectile. You just have to eliminate t from Eqs (4.3 b) and (4.3 d) for x and y. Substituting the value of t from Eq (4.3 b) in Eq (4.3 d) we get

$$y = v_{0y}\frac{x}{v_{0x}} - \frac{1}{2}g\frac{x^2}{v_{0x}} \qquad (4.8\ a)$$

Using Eqs (4.1 a and b), Eq (4.8a) becomes

$$y = (\tan\theta_0)\,x - \frac{g}{2(v_0\cos\theta_0)^2}x^2 \qquad (4.8\ b)$$

This is the equation of a ***parabola.*** You will use this equation in the terminal questions.

Thus, if air resistance is negligible, ***the path of any projectile launched with some horizontal and vertical velocity is a parabola or a portion of a parabola.*** In Fig 4.3 you can see some trajectories of a projectile at different angles of elevation.

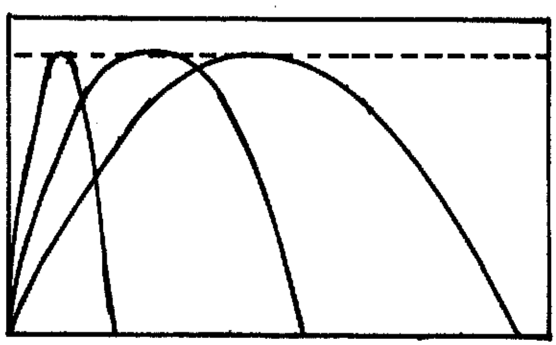

Fig 4.3: *Trajectories of a projectile*

Eqs 4.5 to 4.7 are often handy for solving problems of projectile motion. For example, these equations are used to calculate the launch speed and the angle of elevation required to hit a target at a known range. But you must keep in mind that these equations do not give us a complete description of projectile motion. Now, let us summarise the important equations describing projectile motion launched from a point (x_0, y_0) with a velocity v_0 at an angle of elevation, θ .

Equations of Projectile Motion :

$$a_x = 0 \qquad a_y = -g \qquad (4.9\ a)$$

$$v_x = v_0\cos\theta \qquad v_y = v_0\sin\theta - gt \qquad (4.9\ b)$$

$$x = x_0 + (v_0\cos\theta)t \qquad y = y_0 + (v_0\sin\theta)\,t - \tfrac{1}{2}gt^2 \qquad (4.9\ c)$$

Trajectory,

$$y = y_0 + (\tan\theta)(x - x_0) - \frac{g}{2(v_0\cos\theta)^2}(x - x_0)^2 \qquad (4.9\ d)$$

Notice that these are more general than the ones discussed earlier. The initial coordinates are left unspecified as (x_0, y_0) rather than being placed at (0, 0). Can you derive this general equation of the projectile trajectory? Do it before studying further?

Thus, far you have studied motion of such objects in a plane, which can be put in the category of projectile motion. In projectile motion, the acceleration is constant both in magnitude and direction. There is another kind of two-dimensional motion of interest in which the acceleration is constant in magnitude but not in direction. This is circular motion, which you will now study.

4.4 CIRCULAR MOTION

The motion at constant speed of a phonograph record or a grinding wheel, the moving hands of an ordinary clock, a vehicle turning around a traffic roundabout can all be idealised as examples of circular motion. The movement of gears, pulleys and wheels involves circular motion. The simplest kind of circular motion is uniform circular motion. Examples are a point on a rotating fan blade or a grinding wheel moving at constant speed. One of the most useful applications of uniform circular motion is putting artificial satellites in circular orbits around the earth. We have benefitted immensely from the INSAT series of satellites and other artificial satellites, haven't we? So let us first learn about uniform circular motion.

4.4.1 Uniform Circular Motion

By definition, ***uniform circular motion is motion with constant speed in a circle or along a circular arc.***

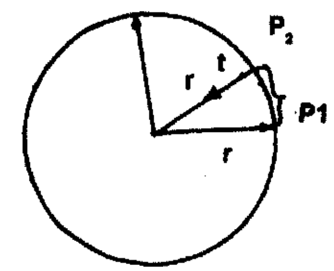

Fig 4.4(a): *Positions of a particle in uniform circular motion.*

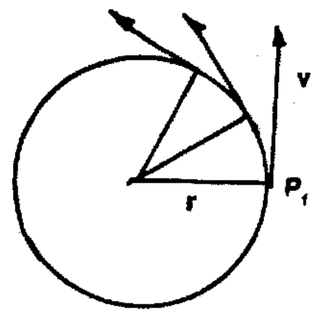

Fig. 4.4(b): *Uniform circular motion*

See Fig 4.4a. It shows the positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of a particle in uniform circular motion at two different times t_1 and t_2, respectively. The word 'uniform' refers to constant speed. We have said that the speed of the particle is constant. What about the particle's velocity? To find out recall the definition of average velocity and apply it to the points P_1 and P_2 for uniform circular motion :

$$\boldsymbol{v}_{av} = \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{t_2 - t_1} = \frac{\Delta \boldsymbol{r}}{\Delta t} \tag{4.10 a}$$

The vector $\Delta\boldsymbol{r}$ is shown in Fig 4.4a. Now suppose you make the time interval Δt smaller and smaller so that it approaches zero. What happens to $\Delta\boldsymbol{r}$? In particular, what is the direction of $\Delta\boldsymbol{r}$? It approaches the tangent to the circle at the point P_1 as Δt tends to zero. Mathematically, we define the instantaneous velocity at the point P_1 as

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} \tag{4.10 b}$$

Thus, for uniform circular motion the magnitude of the velocity, i.e., its speed is constant but the velocity vector is not constant. Can you say why? This is because the velocity (which is tangential to the path) is changing direction continuously as the particle travels around the circle (Fig. 4.5b). Because of ***this change in velocity, uniform circular motion is accelerated motion.*** Our main aim in this section is to determine the acceleration of a particle in uniform circular motion. This is also termed centripetal acceleration for reasons you will learn in a short while.

Centripetal acceleration : Consider Fig (4.5). Let the constant speed of the particle be v and the radius of the circle be r. We have to find the value of acceleration at an instant t, i.e., the instantaneous acceleration of the particle. For this, we have to consider the change in the particle's velocity in an extremely short time interval Δt. In Fig (4.5a) you can see the particle at two positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively, at instants t_1 and t_2. The interval $\Delta t = t_2 - t_1$ is very small. The difference between these position vectors is

$$\Delta \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$$

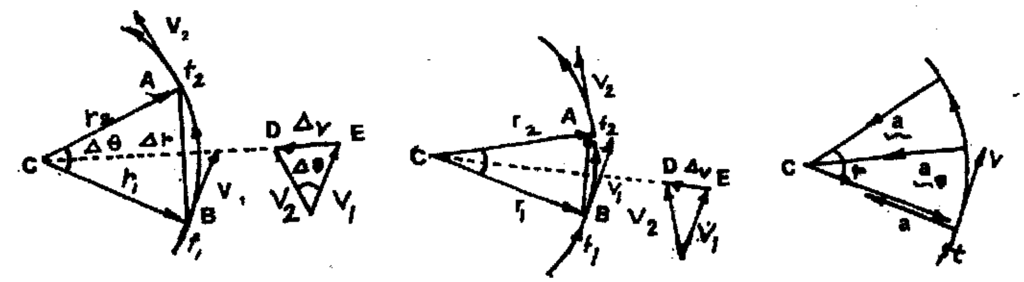

Fig 4.5 *Two positions of particle in circular motion.*

As we decrease t further, $\boldsymbol{r}$ will become smaller (Fig 4.5b). Now look at Figs. 4.5a and b. The velocity vectors at time t_1 and t_2 are now placed tail to tail and their difference is

$$\boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1$$

The angle between the position vectors is θ. Since the velocity vectors are always perpendicular to the position vectors in circular motion, the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is also θ. Since the angles θ in the triangles ABC and DEF are equal, they are similar. Hence, we have

$$\frac{DE}{AB} = \frac{AC}{DF} = \frac{BC}{EF}$$

or

$$\frac{\text{Magnitude of } \boldsymbol{v}}{\text{Magnitude of } \boldsymbol{r}} = \frac{\text{Magnitude of } \boldsymbol{v}}{\text{Magnitude of } \boldsymbol{r}} = \frac{v}{r}$$

We can rewrite this as

$$\text{Magnitude of } \boldsymbol{v} = \frac{v}{r} \times (\text{magnitude of } \boldsymbol{r})$$

Now if Δt is made very small, $\Delta\theta$ will also be very small. Then the straight line segment $\Delta \boldsymbol{r}$ will approximately coincide with the circular arc between points A and B. Thus the arc length is simply equal to the distance travelled by the particle in an extremely small time interval Δt. Therefore, magnitude of $\boldsymbol{v} = v/r \times$ (distance travelled in time Δt).

The magnitude of the acceleration at the instant t is equal to the magnitude of $\Delta \boldsymbol{v}$ divided by the time interval Δt, as we make Δt smaller and smaller making it approach zero. Thus

$$a = \lim_{\Delta t \to 0} \frac{\text{magnitude of } \Delta \boldsymbol{v}}{\Delta t}$$

$$= \frac{v}{r} \times \lim_{\Delta t \to 0} \frac{\text{distance travelled in time } \Delta t}{\Delta t}$$

Now the distance travelled in time Δt divided by the time Δt, (as Δt is made smaller and closer to zero)is just the speed v of the particle.Hence, we have

$$a = \frac{v}{r} \times v$$

or
$$\boxed{a = \frac{v^2}{r}} \tag{4.11a}$$

What is the direction of this acceleration? Study Fig. 4.5b again. For very small Δt the direction of $\Delta \boldsymbol{v}$ will be perpendicular to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (which will be nearly parallel in this limiting case). Hence the instantaneous acceleration is perpendicular to the instantaneous velocity. Since the velocity vector is tangential to the circle, the ***acceleration vector points along the radius, towards the centre of the circle.*** Since the acceleration of the particle is directed towards the centre of the circle, it is called ***centripetal acceleration.*** Thus, whenever you see a particle moving at constant speed v in a circle (or a circular arc) of radius r, you may be sure that it has an acceleration directed toward the centre of the circle, of magnitude v^2/r. To sum up : Particle in uniform circular motion has a centripetal acceleration given by

$$\boldsymbol{a} = -\frac{v^2}{r}\hat{\boldsymbol{r}}, \qquad a = \frac{v^2}{r} \tag{4.11b}$$

where $\hat{\boldsymbol{r}}$ is the unit vector directed from the centre of the circle to the particle (Fig. 4.5c). Notice that the centripetal acceleration given by Eq. (4.11) is changing direction continuously (Fig. 4.5c). Hence, uniform circular motion is not a case of motion with constant acceleration. That is why we cannot use the kinematical equations in this case.

We will now work out an example to give you a feel for the magnitude in real life situations.

Example 4.2: *Astronauts experience high acceleration in their flights in space. In their training at NASA in USA for such situations, they are placed in a closed capsule which is fixed at the end of a revolving arm of radius15 m. The capsule is whirled around a circular path, just like we whirl a stone tied to a string in a horizontal circle. If the arm revolves around the circle at a rate of 24 revolutions per minute, what is the centripetal acceleration of the capsule?*

Solution : The circumference of the circular path is $2\pi\times$(radius)$=2\pi\times15$ m. Since the capsule makes 24 revolutions per minute or 60 s, the time it takes to go once around this circumference is $\frac{60}{24}$ s. Therefore,

the speed of the capsule, $v = \dfrac{2\pi\times15\text{m}}{(60/24)\text{s}} = 38\ \text{ms}^{-1}$.

The magnitude of the centripetal acceleration

$$a = \frac{v^2}{r} = \left(\frac{38\,\text{ms}^{-1}}{15\text{m}}\right) = 96\,\text{ms}^{-2}$$

You may now like to work out some problems to fix these ideas in your mind.

INTEXT QUESTIONS 4.2

1. In uniform circular motion, (a) Is the speed constant? (b) Is the velocity constant? (c) Is the magnitude of the acceleration constant? (d) Is acceleration constant? Explain.

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2. An athlete runs around a circular track with a speed of 9.0 ms^{-1} and a centripetal acceleration of 3 ms^{-2}. What is the radius of the track?

   ....

3. The Fermi lab accelerator is one of the largest particle accelerators. In this accelerator, protons are forced to travel in an evacuated tube in a circular orbit of diameter 2.0 km at a speed nearly equal to the speed of light c (99.99995% of the speed of light). What is the centripetal acceleration of these protons? Take $c = 3\times10^8$ ms^{-1}

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So far you have studied about objects moving in a circle at constant speed. But when you start moving a merry-go-round from rest, the speed of a particle on it varies, until it acquires a constant value. What is the velocity and acceleration of such a particle moving in a circle with variable speed? Let us find out.

4.4.2 Non-Uniform Circular Motion

To deal with such a situation, it is useful to introduce the concepts of angular velocity and angular acceleration. Consider once again a particle in circular motion (Fig. 4.5a). The particle turns by an angle $\Delta\theta$ in time Δt between the two positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. We define the average angular speed of the particle as

$$\omega_{av} = \frac{\theta_2-\theta_1}{t_2-t_1} = \frac{\Delta\theta}{\Delta t} \qquad (4.12\ a)$$

When the angle turned or the angular speed ω_{av} changes with time, we define ***instantaneous angular speed*** as the limit of the ratio $\Delta\theta/\Delta t$ as Δt is made approach zero.

Thus,

$$\omega = \lim_{\Delta t\to0}\frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (4.12\ b)$$

So if we know $\theta(t)$, we can find ω. The units of angular speed are radian per second (rad s^{-1}). If the angular speed of a particle is not constant, then it has an angular acceleration. Let ω_1 and ω_2 be the angular speeds at times t_1 and t_2, respectively. The ***average angular acceleration*** is defined as

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad (4.13\ a)$$

where $\Delta\omega$ is the change in the angular speed in the time interval Δt. The ***instantaneous angular acceleration*** is the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad (4.13\ b)$$

Non-uniform circular motion is actually an example of motion with constant angular acceleration. Let us now determine the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of a particle in circular motion in terms of ω and α, the angular speed and angular acceleration. Consider Fig 4.5a once again. For very small values of $\Delta\theta$, the distance Δr of the particle between the points A and B is related to θ as follows :

$$\Delta r = r\Delta\theta \qquad (4.14)$$

where r is the radius of the circle and it is constant.
The average speed v_{av} is given by

$$v_{av} = \frac{\Delta r}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

Making Δt smaller so that it approaches zero and using Eqs. (4.10 b) and (4.12b) we get

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = r \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

or $$\boxed{v = r\omega} \qquad (4.15)$$

The direction of $\boldsymbol{v}$ is of course along the tangent to the circle. To obtain the acceleration, we first find the average acceleration, i.e.,

$$a_{av} = \frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t}$$

and make Δt approach zero. Thus, using Eq. (4.13 b) we have

$$a_t = r \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = r\alpha \qquad (4.16)$$

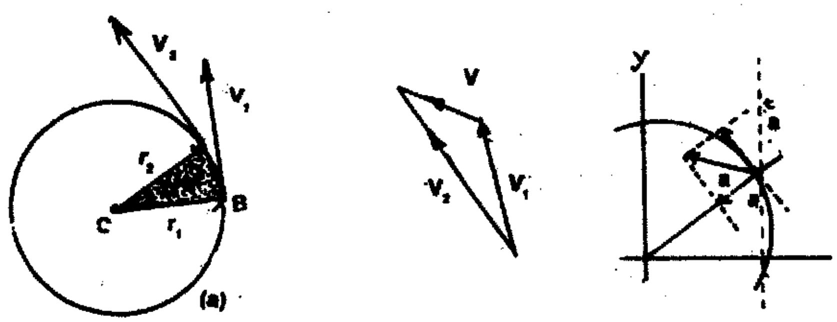


Fig 4.6 : *Particle in non-uniform circular motion.*

This is the component of acceleration which arises when the particle's speed v is changing. The situation now is as follows: A particle is in circular motion such that both the magnitude and direction of its velocity are changing (Fig. 4.5a). The velocity is along the tangent to the circle but now the acceleration has two components – one of these is along the tangent to the circle given by Eq. (4.16) (see Figs. 4.5b and c). This component is called the ***tangential acceleration.***

In addition you will recall that a particle moving in a circular path has a radial component of the acceleration given by v^2/r or $\omega^2 r$ from Eq. (4.15). To sum up, the linear acceleration of a particle moving in a circle has two components.

$$\text{radial acceleration} = a_r = -\frac{v^2}{r}\hat{r} = -\omega^2 r\,\hat{r} \qquad (4.17\text{ a})$$

$$\text{tangential acceleration} = a_t = \alpha\, r\, \hat{n} \qquad (4.17\text{ b})$$

where $\hat{n}$ is along the tangent to the circular path and is positive by convention for anticlockwise motion. Note that a_r is always present as long as angular speed is not zero : a_t is present as long as angular acceleration is not zero.

You should, now, work out some problems to fix these ideas in your mind.

INTEXT QUESTIONS 4.3

1. Consider a point on the edge of a rotating wheel.
   (a) When the wheel rotates with constant angular velocity, does the point have a radial acceleration? Does it have a tangential acceleration?

   ....

   (b) When the wheel rotates with constant angular acceleration, does the point have a radial acceleration? Does it have a tangential acceleration?

   ....

   (c) Do the magnitude of these accelerations change with time in both cases?

   ....

2. The earth's orbit about the sun is nearly circular. What is the angular speed of the earth (taken as a particle) about the sun? What is its linear speed? What is its acceleration with respect to the sun? Assume the radius of the orbit to be 1.5×10^{11} m.

   ....

3. A merry-go-round of radius 2 m starts from rest. Its angular speed increases at a constant rate of 0.1 rad s^{-2}. What tangential acceleration does a child standing at its outer edge have?

   ....

So far you have studied that an object moving in a circle is accelerating. This acceleration has two components – one directed toward the centre of the circle (called ***centripetal or radial acceleration***); other along the tangent (called ***tangential acceleration***). Now you have also studied Newton's laws in lesson 3. From Newton's second law you know that since the object in circular motion is accelerating, a net force must be acting on it.

What is the direction and magnitude of this force? This is what we will find in the next section. Then we will apply Newton's laws of motion to uniform circular motion. This will lead us to answers about why road are banked, or why pilots feel pressed to their seats when they fly aircrafts in vertical loops.

4.5 APPLICATIONS OF UNIFORM CIRCULAR MOTION

Let us first determine the force acting on a particle that keeps it in uniform circular motion. Consider a particle moving with constant speed v in a circle of radius r. From Newton's second law, the net external force acting on a particle is related to its acceleration by

$$\mathbf{F} = m\mathbf{a} \tag{4.18}$$

For a particle in uniform circular motion, a points toward the centre of the circle and its magnitude is $a = v^2/r$. Therefore, from Eq. (4.18), the net external force on the particle must also point toward the centre of the circle.

$$\mathbf{F} = \frac{mv^2}{r}\mathbf{r};\ |\mathbf{F}| = \frac{mv^2}{r} \tag{4.19}$$

This net external force directed toward the centre of the circle with magnitude given by Eq. (4.19) is called the ***centripetal force. An important thing to understand and remember is that the term 'centripetal force' does not refer to a type of force of interaction like the force of gravitation or electrical force.*** This term only tells us that the net force of a certain magnitude acting on a particle in uniform circular motion is directed towards the centre. It does not tell us how this force is provided.

Thus, the force may be provided by the gravitational attraction between two bodies. For example, in the motion of a planet around the sun, the centripetal force is provided by the gravitational force between the two. Similarly, the centripetal force for a car travelling around a bend is provided by the force of friction between the road and the car's tyres and by banking the road. You will understand these ideas better if we apply them to certain concrete examples.

4.5.1 Banking of Roads

Consider a car of mass m travelling with speed v on a curved section of a highway (Fig 4.7). To keep the car moving uniformly in the circular path, a force must act on it. It should be directed towards the centre of the circle and its magnitude must be equal to mv^2/r. Here r is the radius of curvature of the curved section.

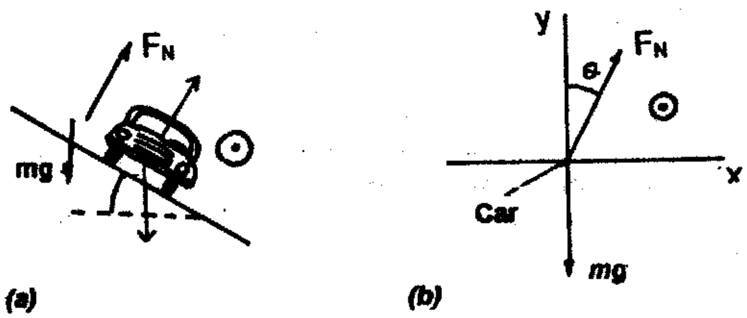


Fig 4.7 : *Banking of roads.*

Now the force of friction between the road and the tyres is often not enough to keep the car in a circular path. Therefore, it is necessary to provide an additional force so that the sum of this force and the force of friction is equal to the centripetal force. Such a force is provided by "banking" of road on curved sections. As a matter of fact, roads are designed to minimise the reliance on friction. For example, when car tyres are smooth or there is water or snow on roads, the coefficient of friction becomes negligible. Roads are banked at curves so that cars can keep on track even when friction is negligible.

What is banking of the road? Banking raises the outer edge of the road above the level of the inner edge (Fig. 4.7a). The angle of banking, θ, is adjusted for the sharpness of the curve and the maximum allowed speed. Let us now analyse the free body diagram for the car to obtain an expression for θ.

Let us consider the case when there is no frictional force acting between the car tyres and the road. The forces acting on the car are the car's weight mg and $\boldsymbol{F}_N$, the force of normal reaction. The centripetal force is provided by the horizontal component of $\boldsymbol{F}_N$. Thus, resolving the force $\boldsymbol{F}_N$ into its horizontal and vertical components, we can write

$$F_N \sin\theta = \frac{mv^2}{r} \qquad (4.20\text{ a})$$

Since there is no vertical acceleration, the vertical component of $\boldsymbol{F}_N$ is equal to the car's weight :

$$F_N \cos\theta = mg \qquad (4.20\text{ b})$$

We have two equations with two unknowns F_N and θ to determine θ, we eliminate F_N from Eqs. (4.20 a and b). Dividing Eq. (4.20 a) by Eq. (4.20 b) we get

$$\tan\theta = \frac{mv^2 / r}{mg} = \frac{v^2}{rg}$$

or

$$\boxed{\theta = \tan^{-1}\frac{v^2}{rg}} \qquad (4.21)$$

How do we interpret Eq. (4.21) for limits on v and choice of θ? Firstly, Eq. (4.21) tells us that the angle of banking is independent of the mass of the vehicle. So even large trucks and other heavy vehicles can ply on banked roads.

Secondly, θ should be greater for high speeds and sharp curves (i.e., for lower values of r). For a given θ, if the speed is less than v given by Eq. (4.21), the car will tend to slide down the incline, i.e., move towards the inner edge of the curved road. And if the speed is more than v, it will tend to slide up, i.e., move towards the outer edge of the curved road. So a vehicle driver must drive within prescribed speed limits on curves. Otherwise, the vehicle will be pushed off the road and there may be accidents.

Usually, due to frictional forces, there is a range of speeds on either side of v. Vehicles can maintain a stable circular path around curves if their speed remains within this range. To get a feel of actual numbers, consider the design of a racetrack of radius 300m which has to allow for speeds upto 50 ms^{-1}. What should the angle of banking be? You may like to quickly use Eq. (4.21) and calculate θ.

$$\theta = \tan^{-1}\frac{(50\,ms^{-1})^2}{(300\,m)\,(9.81\,ms^{-2})} = \tan^{-1}$$

You may like to consider another application.

4.5.2 Aircrafts in vertical loops

On Republic Day and other shows by the Indian Air Force you may have seen pilots flying aircrafts in vertical loops (Fig. 4.8a). In such situations, at the bottom of the loop, the pilots feel as if they are being pressed to their seats by a force of gravity equal to several *g*'s. Let us understand why this happens. Fig. 4.8 b shows the 'free body' diagram for the pilot at the bottom of the loop. The forces acting on him are *mg* and the normal force ***N*** exerted by the seat. The net vertically upward force is ***N*** *mg* and this provides the centripetal acceleration :

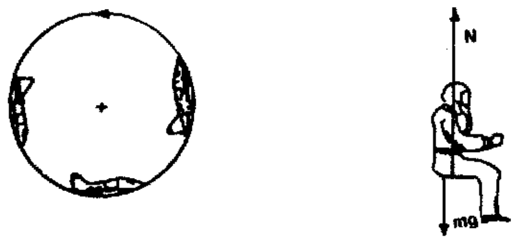


Fig 4.8 : *(a) Aircrafts in vertical loops (b) Free-body diagram for a pilot*

$$N - mg = ma = mv^2/r$$

$$\text{or } N = m\,(g + v^2/r)$$

In actual situations, v = 200 m s^{-1} and r = 1500 m which gives

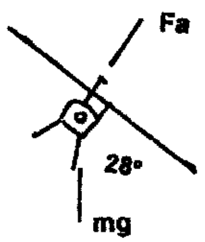
$$N = mg\left[1 + \frac{(200\,ms^{-1})^2}{9.81\,ms^{-2} \times 1500\,m}\right] = mg \times 3.7$$

So the pilots feel as though force of gravity has been magnified by a factor of 3.7 and say that they are experiencing 3.7 *g* force. If this force exceeds set limits, pilots may even black out for a while and it could be dangerous.

You may now like to apply these ideas to some other situations.

INTEXT QUESTIONS 4.4

1. Aircrafts are usually banked while taking a turn when flying at a constant speed (Fig 4.8). ***Fa*** is the force exerted by the air on the aircraft. Explain why aircracts are banked. Draw a freebody diagram for this aircraft. Suppose an aircraft travelling at a speed v = 100 ms^{-1} makes a turn at a banking angle of 30°. What is the radius of curvature of the turn? Take g = 10 ms^{-2}.



....

2. Calculate the maximum speed of a car which makes a turn of radius 100 m on a horizontal road. The coefficient of friction between the tyres and the road is 0.90. Take g = 10 ms^{-2}.

....

3. An interesting act performed at variety shows is to swing a bucket of water in a vertical circle such that water does not pour out while the bucket is inverted at the top of the circle. For this trick to be performed successfully, the speed of the bucket must be larger than a certain minimum value. Determine an expression for the minimum speed of the bucket at the top of the circle in terms of its radius R. Calculate the speed for $R = 1.0$ m.

....

So far you have studied about motion from the point of view of inertial observers. Do you think we are all inertial observers? Remember that the earth is rotating on its axis, i.e., any object on it accelerates. Thus, any frame of reference attached to the earth is non-inertial. So for an accurate description of phenomena occurring on the earth we need to use a non-inertial frame of reference. You can find several examples of non-inertial frames in your immediate environment. For example, a rotating merry-go-round, a bus accelerating from rest, a car rounding the corner and a ball falling freely.

4.6 NON-INERTIAL FRAMES OF REFERENCE

Suppose you are standing on a roadside observing the motion of two buses, A and B. Bus A moves with a constant velocity and Bus B accelerates with respect to you. Then the frames of reference attached to you and Bus A are inertial frames of reference with respect to each other. And the frames of reference attached to you and Bus B are non-inertial frames of reference with respect to each other. In general, ***the frames of reference moving with uniform velocity with respect to each other are inertial and those accelerating with respect to each other are non-inertial.***

Imagine you are sitting on a bench in a park and observing these activities going around you :

(i) A child sitting on a rotating merry-go-round;
(ii) A ball thrown up in the air
(iii) An elderly man sitting on another bench in a park
(iv) A youth walking leisurely (with a low uniform speed)

Suppose you attach a frame of reference with the child, the ball, the elderly man and the youth mentioned in (i) to (iv). Can you identify which of these frames are inertial and which ones non-inertial with respect to you? Clearly, the frames of reference in (i) and (ii) are non-inertial as they are accelerating. Now that you know what is meant by a non-inertial frame of reference.

INTEXT QUESTIONS 4.5

1. A glass half filled with water is kept on a horizontal table in a train. Will the free surface of water remain horizontal as the train starts?

....

2. When a car is driven too fast around a curve it skids outward. How would a passenger sitting inside explain the car's motion? How would an observer standing on a road explain the event?

....

3. A tiny virus of particle mass 6×10^{-19} kg is in a water suspension in a centrifuge being rotated at an angular speed of $2\pi \times 10^3$ rad s^{-1}. The particle is at a distance of 4 cm from the axis of rotation. Calculate the net centrifugal force acting on the particle.

....

4. What must the annular speed of the earth be so that the centrifugal force makes objects fly off its surface? Take $g = 10$ ms^{-2}

....

Let us now summarise the concepts presented in this lesson.

4.7 WHAT YOU HAVE LEARNT?

- **Projectile motion** is defined as motion which has constant velocity in a certain direction and constant acceleration in a direction perpendicular to that of velocity.

$a_x = o$	$a_y = -g$
$v_x = v_0 \cos\theta$	$v_y = v_0 \sin\theta - gt$
$x = x_0 + (v_0 \cos\theta)t$	$y = y_0 + (v_0 \sin\theta) - ½ gt^2$

- Height $$R = \frac{v_0^2 \sin 2\theta}{g}$$

- Time of Flight $$T = \frac{2v_0 \sin\theta}{g}$$

- Range $$R = \frac{v_0^2 \sin 2\theta}{g}$$

- Trajectory $$y = (\tan\theta_0) \times -\frac{g}{2\left(v_0 \cos\theta_0\right)^2} \times 2$$

- ***Circular motion***, i.e., motion in a circle or along a circular arc, is uniform when the speed of the particle is constant. A particle undergoing ***uniform circular motion*** in a circle of radius r at constant speed v has a ***centripetal acceleration*** given by

$$a_r = -\frac{v^2}{r} r$$

where r is the unit vector directed from the centre of the circle to the particle. The speed v of the particle is related to its angular speed ω by $v = r\omega$

- The ***centripetal force*** acting on the particle is given by

$$F = ma_r = -\frac{mv^2}{r}\hat{r}$$

- A particle executing non-uniform circular motion has a centripetal acceleration a_r as well as a ***tangential acceleration*** a_t given by $a_t = \alpha r \hat{n}$ where α is the angular acceleration and n is a unit vector, tangent to the circle.

- ***Frames of reference*** moving with uniform velocity with respect to each other are ***inertial*** and the frames accelerating with respect to each other are ***non-inertial.***

- A body of mass m observed from a non-inertial frame of reference undergoing acceleration a is acted upon by an ***inertial force*** F given by $F = -ma$

- Observed from a rotating frame of reference, a particle of mass m is acted upon by an inertial force called the ***centrifugal force*** $F_{cent} = m\omega^2 r \hat{r}$ where ω is the angular speed of the rotating frame and r the unit vector directed from the origin of the frame to the particle.

4.8 TERMINAL QUESTIONS

1. Why does a cyclist bend inward while taking a turn on a circular path?

2. Explain why the outer rail is raised with respect to the inner on the curved portion of a railway track?

3. If a particle is having circular motion with constant speed, will its acceleration also be constant?

4. A stone is thrown from the window of a bus moving on horizontal road. What path will the stone follow while reaching the ground?

5. A hunter aims exactly at a monkey sitting on a tree. As soon as the hunter fires the shot, the monkey jumps from the tree. Will the monkey be shot? What would happen, if the moneky had not jumped?

6. A string can sustain a maximum force of 100N without breaking. A mass of 1 kg is tied to one end of the piece of string of 1m long and it is rotated in a horizontal plane. Compute the maximum speed with which the body can be rotated without breaking the string?

7. A motorcyclist passes a curve of radius 50 m with a speed of 10 ms^{-1}. What will be the centrepetal acceleration when turning the curve?

8. A bullet is fired with an initial velocity 300 ms^{-1} at an angle of 30° with the horizontal. At what distance from the gun will the bullet strike the ground?

9. The length of the second's hand of a clock is 10 cm. What is the speed of the tip of this hand?

10. You must have seen heroes in Hindi films jumping huge gaps on horse backs and motor cycles. In this problem consider a daredevil motor cycle rider trying to cross a gap at a velocity of 100 km h^{-1}. (Fig 4.9) Let the angle of incline on either side be 45°. Calculate the widest gap he can cross.

11. A shell is fired at an angle of elevation of 30° with a velocity of 500 ms^{-1}. Calculate the vertical and horizontal components of the velocity, the maximum height that the shell reaches, and its range.

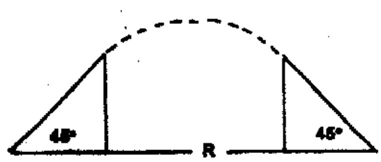


Fig 4.9

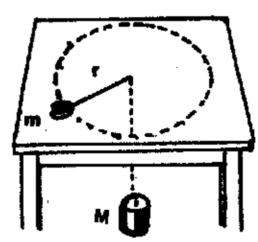


Fig 4.10

12. An aeroplane drops a food packet from a height of 2000 m above the ground while in horizontal flight at a constant speed of 200 km h^{-1}. How long does the packet take to fall to the ground? How far ahead (horizontally) of the point of release does the packet land?

13. A mass m moving in a circle at speed v on a frictionless table is attached to a hanging mass *M* by a string through a hole in the table (Fig 4.10). Determine the speed of mass *m* for which the mass *M* would remain at rest.

14. A car is rounding a curve of radius 220 m at a speed of 60 km-h^{-1}. What is the centrifugal force on a passenger of mass m = 85 kg?

4.9 ANSWERS TO THE INTEXT QUESTIONS

Intext Questions 4.1

(1) (a), (b), (d)

(2) (ii), (v)

(3) (a) Yes (b) Yes
(c) The ball with the maximum range.

(4) Maximum Range =

$$\frac{v_0^2}{g} = \frac{(9.5\ ms^{-1})^2}{9.78 ms^{-2}} = 9.23 m$$

Thus, the difference is 9.23 m – 8.90 m = 0.33 m.

Intext Questions 4.2

(1) (a) Yes (b) No (c) Yes (d) No
The velocity and acceleration are not constant because their directions are changing continuously.

(2) Since

$$a = \frac{v^2}{r}, r = \frac{v^2}{a} = \frac{(9.0 ms^{-1})^2}{3ms^{-2}} = 27 m$$

(3) $$a = \frac{c^2}{r} = \frac{(3\times10^8\ ms^{-1})^2}{1.0\times10^3\ m} = 9\times10^{13} ms^{-2}$$

Intext Questions 4.3

(1) (a) The point on the wheel has a radial acceleration but no tangential acceleration as its angular speed is constant. (b) Now it has both radial and tangential accelerations. (c) In case (a) the magnitude of radial acceleration is constant. In case (b) the magnitude of radial acceleration is not constant but the magnitude of tangential acceleration is constant.

(2) The earth moves 2π rad angles in 365 days around the sun. Therefore its angular speed

$$\omega = \frac{2p\ rad}{365\times24\times60\times60 s} = 2\times10^{-7} rad s^{-1}.$$

Linear speed
$v = \omega r = 2 \times 10^{-7}$ rad $s^{-1} \times 1.5 \times 10^{11}$ m
$= 3.0 \times 10^4$ m s^{-1}
Acceleration with respect to the sun

$$= \frac{v^2}{r} = \left(\frac{3.0\times10^4\ ms^{-1}}{1.5\times10^{11}\ m}\right)^2 = 6\times10^{-3} r$$

(3) The angular acceleration is α = 0.1 rad s^{-2}. Thus the tangential acceleration of the child is
$a_t = \alpha r = 0.1$ rad $s^{-2} \times 2m = 0.2$ m s^{-2}

Intext Questions 4.4

(1) This is similar to the case of banking of roads. If the aircraft banks, there is a component of the force F_a exerted by the air along the radius of the circle to provide the centripetal acceleration. Fig. 4.11 shows the free body diagram. The radius of curvature is

$$R = \frac{v^2}{g\tan} = \left(\frac{100 ms^{-1}}{10 ms^{-2} \times \tan 30°}\right)^2$$

$$= 10\sqrt{3} m$$
$$= 17.3 m.$$

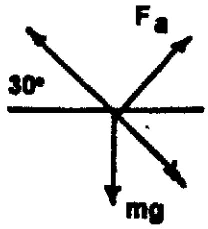


Fig. 4.11

(2) The force of friction provides the centripetal acceleration :

$$F_s = \mu_s N = \frac{mv^2}{r}$$

Since the road is horizontal N = mg

Thus $\mu_s mg = \frac{mv^2}{r}$

or $v^2 = \mu_s g r$
or $v = (0.9 \times 10\ m\ s^{-2} \times 100\ m)^{½}$
$v = 30\ ms^{-1}$.

(3) Refer to Fig. 4.12 showing the free body diagram for the bucket at the top of the circle. In order that water in the bucket does not fall but keeps moving in the circle, the force mg should provide the centripetal acceleration. At the top of the circle.

$$mg = \frac{mv^2}{R}$$

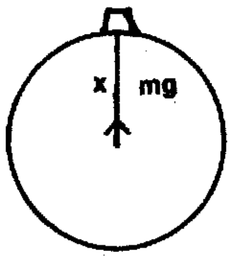


Fig 4.12

or $v^2 = Rg$

$\therefore v = \sqrt{Rg}$

This is the minimum value of the bucket's speed at the top of the vertical circle. For $R = 1.0$ m and taking $g = 10\ ms^{-2}$ we get
$v = 10\ m\ s^{-1} = 3.2\ ms^{-1}$

Intext Question 4.5

(1) When the train starts it has an acceleration, say **a**. Thus the total force acting on water in the frame of reference attached to the train is
$F = mg - ma$

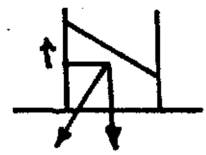


Fig 4. 13

where m is the mass of the water and the glass. (Fig 4.13). The surface of the water takes up a position normal to F as shown.

(2) To the passenger sitting inside, a centrifugal force $(=mv^2/r)$ acts on the car. The greater v is the larger r would be. To an observer standing on the road, the car moving in a curve has a centripetal acceleration given by mv^2/r. Once again, the greater v is, the larger r will be.

(3) The net centrifugal force on the particle is $F = m\omega^2 r = (6 \times 10^{-19}\ kg) \times (2\pi \times 10^3\ rad\ s^{-1})^2 \times (0.04\ m) = 9.6 \times 10^{-13}$N.

ANSWERS TO THE TERMINAL QUESTIONS

6. $10\ ms^{-1}$
7. $2\ ms^{-2}$
8. 8.83×10^3 m
9. $\pi/3\ cms^{-1}$.
10. Let us first convert the velocity to units of m s^{-1} .

$$100\,kmh^{-1} = \frac{100 \times 1000}{60 \times 60}\,ms^{-1}$$

$= 27.8\ ms^{-1}$

$= 28\ ms^{-1}$

Horizontal component of velocity
$V_{xi} = 28 \cos 45°\ ms^{-1}$
$= 19.8\ ms^{-1}$
$= 20\ ms^{-1}$

Vertical component of velocity
$V_{yi} = 19.8\ ms^{-1}$
$= 20\ ms^{-1}$

For the widest gap the trajectory would like the one shown in Fig. 4.1. The motor cycle comes down to its original height as it crosses the gap. This means that in the time it travels the distance R, it returns to the same height h = 0. Now, using the equation for y component of distance
$h = V_{yi} t + ½ gt^2$

or $0 = (20\ m\,s^{-1})t + ½\,(-10\ ms^{-2}) \times t^2$
or $t = 4$ s.
Thus $R = V_x t = 20\ ms^{-1} \times 4$ s
$= 80$ m.
Thus the widest gap is 80 m.

11. The vertical component of the shell's velocity
$= (500\ ms^{-1}) \cos 30°$
$= 250\ 3\ ms^{-1}$
The horizontal component of the shell's velocity
$= (500\ ms^{-1}) \sin 30°$
$= 250\ ms^{-1}$
Maximum height of the shell

$$= \frac{\left(500\,ms^{-1}\right)^2 \times \sin^2 30°}{2 \times 10\,ms^{-2}}$$

$= 3125$ m.
Range of the shell

$$= \frac{\left(500\,ms^{-1}\right)^2 \sin 60°}{10\,ms^{-2}}$$

$= 12500\ \ 3$ m.

12. In this case the height is 2000 m. The packet is dropped at a horizontal speed of 200 km h^{-1} or (200 × 1000)/(60 × 60) m s^{-1}, 55.6 ms^{-1}. To find the time of flight we use the equation

$$y - y_o = (v_o \sin \theta_o)t - \frac{1}{2} gt^2$$

Putting $y - y_o = -2000$ m (the minus sign occurs because the origin is taken at the aircraft) and

$\theta_o = 0$ we get

$$-2000 \text{ m} = -\frac{1}{2} gt^2$$

or $t^2 = 400$ s ($g = 10$ ms^{-2})

$t = 20$ s.

The horizontal distance covered by the packet is given by

$$x - x_o = (_o \cos \theta_o)t$$

$$= (55.6 \text{ ms}^{-1}) \times 20 \text{ s.}$$

$$= 1112 \text{ m.}$$

13. The mass M would remain at rest if the net external force on it is zero. There are two force on the mass M, Mg and **T**, the tension in the string. If the string is considered massless, **T** is also the tension in the string connecting m. This provides the centripetal force for m to move in a circle. Thus,

$$T = Mg$$

and $T = mv^2/r$

or $Mg = mv^2/r$

$$\therefore v^2 = \frac{M}{m} rg$$

$$\text{or } v = \sqrt{\frac{Mrg}{m}}$$

14. The centrifugal force on the car is

$$F_{cent} = \frac{mv^2}{r} r$$

$$\text{or } F_{cent} = \frac{85 \text{ kg} \times (60{,}000)^2}{220 \text{ m} \times (3600)^2} \text{N}$$

$$= 107 \text{ N.}$$