7

ROTATIONAL MOTION

7.1 INTRODUCTION

So far you have studied the motion of particles under gravitational and other forces. ***Did the equation of motion of the particle involve its mass?*** The reason that the size of the particle does not appear in its equation of motion is that we idealized the particles as ***point particles*** having only mass and no size. Such particles are also called ***mass points.*** In real life there are no such particles. The bodies that we have to deal with, have large number of particles. Even a tiny marble consists of millions of particles. However, in the previous lessons we have seen that the size of a body has not been taken in to account. For example the size of a planet revolving around the sun has not been considered and it was taken just a point particle.

However, when we have to study the rotation of a body, the finite size of the body cannot be ignored. For example, when we consider phenomena connected with the rotation of the body on its own axis, we do take note of its finite size.

In this lesson we will study the ***rotational motion*** of bodies, so their finite sizes become important for us.

7.2 OBJECTIVES

After studying this lesson, you should be able to,

- *define a rigid body and realize that it is an idealization;*
- *define the centre of mass of a rigid body and recognize its importance;*
- *recognize that the general motion of a rigid body consists of both translational motion and rotational motion;*
- *define moment of inertia and realize its physical significance;*
- *state theorems of parallel and perpendicular axes and apply them;*
- *define torque and compute the direction of rotation produced by a torque*
- *define angular momentum and write down the equation of motion of a rigid body;*
- *state the principle of conservation of angular momentum and cite a few examples in support of this principle;* and
- *calculate the velocity acquired by a rigid body at the end of its motion on an inclined plane.*

7.3 RIGID BODY

We have noted above that point particles or point masses are idealizations. In real life we meet systems or bodies which consist of a large number of particles. We have also noted that when these systems interact with other systems which are at distances very much larger compared to their sizes, then their sizes can be ignored. *Can you give two examples of such cases where the sizes of the bodies are not important?*

But when we have to consider the rotation of a body about an axis, then the body has to be considered as a whole and the size of the body becomes important.

When we consider the rotation of a system, we generally assume that during rotation the distances between its constituent particles remain fixed. Such a system of particles is called a ***rigid body.***

We define a rigid body as one in which the distances between the particles do not change as the body moves about.

The above definition implies that the shape of a rigid body is preserved during its motion. However, like a point particle a rigid body is also an idealization, because if we apply large pressures the distances between the particles do change, may be by very small amounts. Therefore, in nature there is nothing like a really rigid body. For most purposes, a solid body is a good enough approximation to a rigid body. A cricket ball, a wooden block, a steel disc, even the earth and the moon would all be considered as rigid bodies in this lesson. *Could a bob of plasticine be considered a rigid body?*

Now, let us check what you have understood about rigid body.

INTEXT QUESTIONS 7.1

1. A frame is made of six wooden rods. The rods are attached to each other in such a way that they cannot move. Can this system be considered a rigid body?

   ....

2. If the distances between the particles of a body do not remain fixed as it moves about, what would be the nature of paths described by the particles?

   ....

7.4 CENTRE OF MASS (C.M.) OF A RIGID BODY

Before we deal with rigid bodies consisting of several particles, let us consider a much simpler case. Suppose we have a system of two particles of equal masses joined by a rod which has no mass and whose length remains fixed. *Can we consider this system of two particles a rigid body?*

Suppose that the two particles are at heights z_1 and z_2 from a horizontal surface (Fig 7.1).

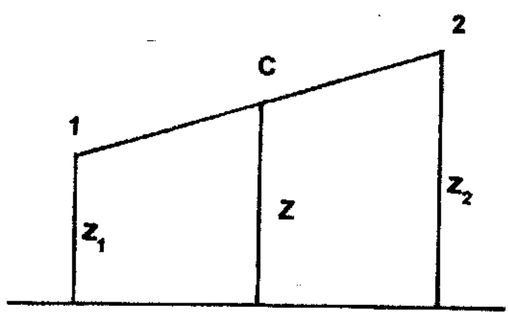


Fig 7.1 : *Two particle system.*

Suppose, further that the gravitational force is uniform in the small region in which the two particles move about. The force on particle 1 is *mg*, and the force on the other particle is also *mg*. The total force acting on the system is therefore 2 *mg*. The problem now is to find a point somewhere in the system so that if a force 2 *mg* acts at that point, the motion of the system would be the same as with two forces, *mg* each, acting on the two particles.

The potential energy of particle 1 is mgz_1 and that of particle 2 is mgz_2. Suppose the force 2 *mg* acts at the point C at a height *z* from the horizontal surface. Since this must be equal to the combined potential energy of the two particles, we have

$$2\,mgz = mgz_1 + mgz_2 \qquad (7.1)$$

$$\text{or} \quad z = \frac{z_1 + z_2}{2} \qquad (7.2)$$

In this case the point 0 lies midway between the two particles. If the two masses are unequal then this point will not be in the middle. If the mass of particle 1 is m_1 and that of particle 2 is m_2 then instead of (7.1) we have

$$(m_1 + m_2)gz = m_1 gz_1 + m_2 gz_2 \qquad (7.3)$$

so that

$$\boxed{z = \frac{m_1 z_1 + m_2 z_2}{(m_1 + m_2)}} \qquad (7.4)$$

The point C is called the **centre of mass** (CM).

Suppose that the mass of the particle is twice that of the other, find the location of the CM.

When a body consists of several particles, then we generalize (7.4) to define its CM. If the particle with mass m_1 has coordinates (x_1, y_1, z_1) with respect to some coordinate frame, mass m_2 has coordinates (x_2, y_2, z_2), and so on (Fig 7.2), then the coordinates of CM are given by

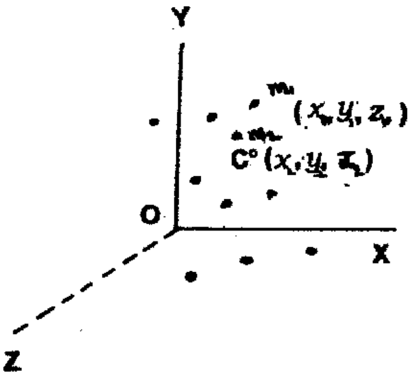


Fig 7.2 : C.M. of a body consisting several particles.

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots\ldots}{m_1 + m_2 + \ldots\ldots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$= \frac{\sum_i m_i x_i}{M} \qquad (7.5)$$

$$Y = \frac{\sum_i m_i y_i}{M} \tag{7.6}$$

$$Z = \frac{\sum_i m_i z_i}{M} \tag{7.7}$$

where $\sum_i m_i$ denotes the sum over all particles and, therefore, $\sum_i m_i = M$ the total mass of the body. Why did we go to such length to define CM? Recall that the rate of change of displacement is velocity, and the rate of change of velocity is acceleration. If a_{1x} denotes the acceleration of particle 1 along the x-axis, then from (7.5) we could write

$$Ma_x = m_1 a_{1x} + m_2 a_{2x} + \ldots\ldots \tag{7.8}$$

where a_x is the acceleration of the centre of mass along the x-axis. Similar equation can be written for accelerations along y - and z - axis. These equations can, however, be combined into a single equation using vector notation. Instead of (7.8) we have

$$M\boldsymbol{a} = m_1 \boldsymbol{a}_1 + m_2 \boldsymbol{a}_2 + \ldots\ldots\ldots \tag{7.9}$$

But the product of mass and acceleration is force. $m_1 \boldsymbol{a}_1$ is therefore the sum of all forces acting on particle 1. Similarly, $m_2 \boldsymbol{a}_2$ gives the net force acting on particle 2. The right hand side is, thus, the total force acting on the body. The forces acting on a body can be of two kinds. Some forces can be due to sources outside the body. These forces are called the ***external*** forces. A familiar example is the force of gravity. Some other forces arise due to the interaction among the particles of the body. These are called ***internal*** forces. Normally in the case of a rigid body, the sum of the internal forces is zero. Therefore, the accelerations of the individual particles of the body are due to the sum, or resultant, of the external forces. In the light of this, we may write (7.9) in the form

$$M\boldsymbol{a} = \boldsymbol{F}_{ext} \tag{7.10}$$

This shows that ***the CM of a body moves as though all the mass of the body were concentrated at that point and it was acted upon by the sum of all the external forces acting on the body.*** The fact that the motion of the CM is determined by the external forces and that the internal forces have no role in this at all leads to very interesting consequences.

You are already familiar with the motion of a body thrown at an angle to the vertical, the motion of the projectile. *Do you recall what this motion is like?*

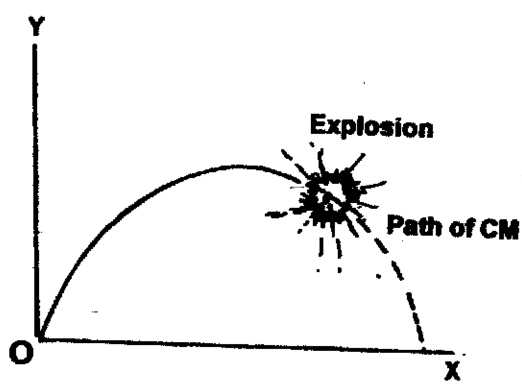


Fig 7.3 : *Centre of mass of a projectile.*

Suppose the projectile is a bomb which explodes in midair and breaks up into several fragments. The explosion is caused by the internal forces. There is no change in the external force, which is the force of gravity. The centre of mass of the projectile, therefore, continues on the parabolic path (Fig 7.3) which the bomb would have described if it had not exploded.

Have you noticed how important is the centre of mass of a rigid body? More of this importance you are going to see very soon. Let us, therefore, see how the centre of mass of a system can be found by taking a simple example.

Example 7.1: *Suppose four masses, 1.0 kg, 2.0 kg, 3.0 kg and 4.0 kg are located at the corners of a square whose each side is 1.0 m. Where would be its centre of mass?*

Solution: We can always make the square lie in a plane. Let this plane be the (x, y) plane. Further, let one of the corners fall at the origin and the sides are the x – y axes. Then the coordinates of the four masses are : m_1 (0,0), m_2 (1.0, 0), m_3 (1.0, 1.0) and m_4 (0, 1.0) where all distances are in metres (Fig 7.4).

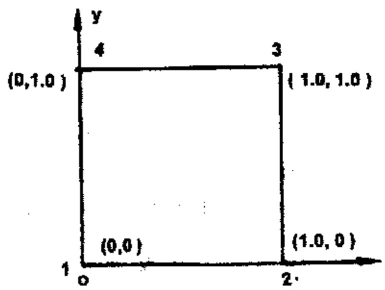


Fig 7.4

Form (7.5) and (7.6), we get,

$$X = \frac{1.0 \times 0 + 2.0 \times 1.0 + 3.0 \times 1.0 + 4.0 \times 0}{1.0 + 2.0 + 3.0 + 4.0} \text{m}$$

$$= 0.5 \text{ m}$$

$$Y = \frac{1.0 \times 0 + 2.0 \times 0 + 3.0 \times 1.0 + 4.0 \times 1.0}{1.0 + 2.0 + 3.0 + 4.0} \text{m}$$

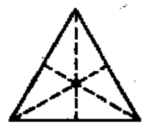
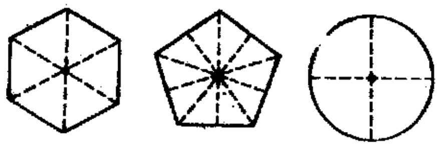
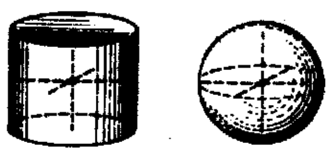
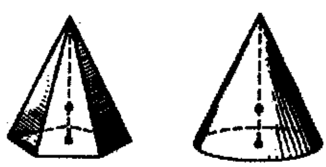
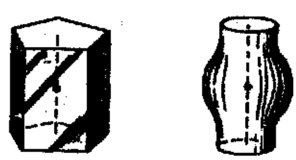
$$= 0.7 \text{ m}$$

The CM has coordinates (0.5 m, 0.7 m) and is marked C in Fig 7.4. We notice that the CM is not at the centre of the square although the square is a symmetrical figure. *What could be the reason for the CM not being at the centre?*

C.M. of Some Bodies

Was it not simple to calculate the position of the CM? But unfortunately it is true only for very simple systems. The bodies that we have to deal with have very large number of particles and this simple method does not work. The computation of the CM of such bodies is a complicated affair. The fact that the masses of all the particles of a rigid body are equal, makes things somewhat simpler. If the body is regular in shape and possesses some symmetry, say it is cylindrical or spherical, then the calculation is a little bit more simplified. But even then these calculations are beyond the scope of this unit. But keeping in mind the importance of CM, we give in Table 7.1 the position of the CM of some regular and symmetric bodies.

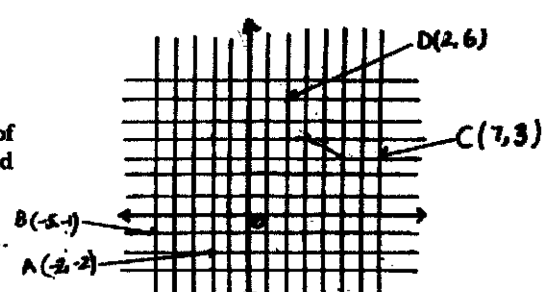
Table 7.1 : *Centres of Mass of some regular and symmetric bodies*

Figure	Position of Centre of Mass
	*Triangular plate* Point of intersection of the three medians
	*Regular polygon and circular plate* At the geometrical center of the figure
	*Cylinder and sphere* At the geometrical center of the figure
	*Pyramid and cone* On line joining vertex with center of base and at ¼ of the length measured from the base.
	*Figure with axial symmetry* Some point on the axis of symmetry
	*Figure with center of symmetry* At the center of symmetry

Now, it is time to check your progress.

INTEXT QUESTIONS 7.2

1. The grid shown here has particles A, B, C, D and E of masses 1.0 kg, 2.0 kg, 3.0 kg, 4.0 kg and 5.0 kg. Find the centre of mass of the system.



....

2. If three particles of masses $m_1 = 1$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg are situate at the corners of an equilateral triangle of side 1.0 metre, find the centre of mass of the system.

....

3. Show that the distances of the two particles from their centre of mass is inversely proportional to their masses.

....

7.5 ROTATIONAL MOTION OF A RIGID BODY

You must have played a game in which you toss a ball to a friend and the friend tosses it back to you (Fig 7.5). Play this game once again, but this time mark some points on the surface of the ball. Notice the path traced out by each point. *How are the paths traced by different points related to one another?*

Fig 7.5: Motion of a ball

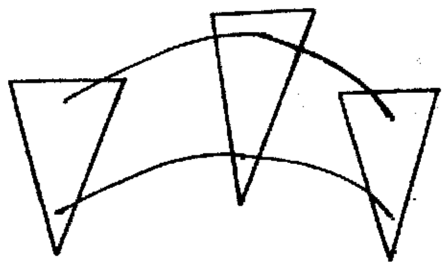


Fig 7.6: Motion of a rigid body.

If a rigid body moves in such a way that all its particles move along parallel paths (Fig 7.6), then its motion is called **translational** motion. The motion of all the particles being identical, the centre of mass must also be tracing out an identical path. Therefore, the translational motion of the body may be represented by the motion of its centre of mass. We have seen that this motion is given by (7.10),

$$M\mathbf{a} = \mathbf{F}_{ext}$$

Do you now see the advantage of defining a centre of mass of a body? With its help the translational motion of a body can be described by an equation similar to that of single particle. The particle here has mass equal to the mass of the whole body. It is located at the centre of mass and is acted upon by the sum of all the external forces which are acting on the rigid body.

Have you ever seen in slow motion the ball delivered by a spin bowler in the game of cricket? If you have, then you must have seen that the ball rotates while moving forward. If you have not seen this, you can perform a little experiment of your own. With your friend play the game of tossing the ball, as you did earlier. But this time give the ball a twist or spin before tossing to your friend. Notice the movements of the points on the ball carefully. This time the points on the ball would undergo looping motion, something like shown in Fig 7.7. This shows that the general motion of a rigid body is a combination of translation and rotation. The earth also performs these two motions at the same time. It rotates on its axis while advancing it its orbit round the sun.

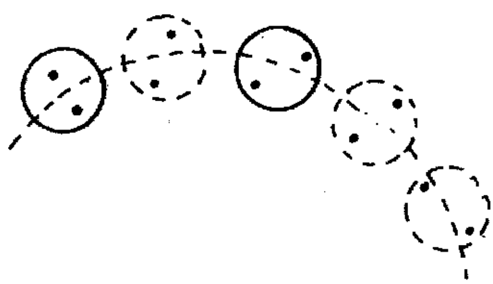


Fig 7.7

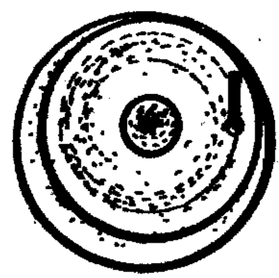


Fig 7.8

While the general motion of a rigid body consists of both translation and rotation, if one point in the body is fixed, it cannot have translational motion. It can then only rotate. The most convenient point for this purpose is the CM of the body.

You must have seen a grinding stone (the chakki). The handle of the stone moves in a circular path. All the points on the stone also move in circular paths round an axis passing through the centre of the stone (Fig 7.8).

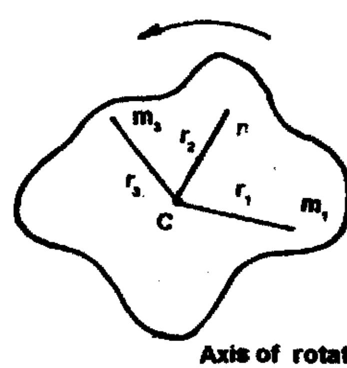
The motion of a rigid body in which all the particles of the body describe circular paths is known the rotational motion.

We have noted above that the translational motion of a rigid body can be described by an equation similar to that of a single particle. You are already familiar with such equations. Therefore, in this lesson we concentrate only on the rotational motion of a rigid body. The rotational motion can be obtained by fixing a point of the body. For the sake of convenience this point can be the CM. The rotation is then about an axis passing through the CM.

In the linear motion of a body that you studied in earlier lessons, the mass of the body plays a very important role. It determines the acceleration acquired by the body for a given force. Is there a similar quantity for the rotational motion too? Let us find out?

7.5.1 Moment of Inertia

Let C be the centre of mass of the rigid body and let the body rotate about an axis through this point (Fig 7.9). Imagine a particle of mass m_1 at a distance of r_1 from the axis of rotation. Let v denote the magnitude of its velocity. Then its kinetic energy is $\frac{1}{2} m_1 v_1^2$. Similarly the kinetic energy of another particle of mass m_2 is $\frac{1}{2} m_2 v_2^2$. If we add the kinetic energies of all the particles, then we get the total energy of the body. If T denotes the total kinetic energy of the body then



Axis of rotation ⊥ to the plane of the paper

Fig 7.9 : *A rigid body rotating about an axis.*

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + ...$$

$$= \sum_i \frac{1}{2} m_i v_i^2 \qquad (7.11)$$

where $\sum_i$ indicates, as before, the sum over all the particles of the body.

You have learnt in earlier lessons the relation between the linear velocity and the angular velocity. The angular velocity is usually denoted by ω. *What is the relation between v and ω ?*

Using this relation in (7.11), we get

$$T = \sum_i \frac{1}{2} m_i (r_i \omega)^2 \qquad (7.12)$$

Since all the particles have the same angular velocity, therefore ω is same for all of them. This equation may be rewritten as,

$$T = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

$$= \frac{1}{2} I \omega^2 \qquad (7.13)$$

The quantity

$$\boxed{I = \sum_i m_i r_i^2} \qquad (7.14)$$

is called the ***moment of inertia*** of the body.

It is important to remember that the ***moment of inertia*** is defined with reference to the axis of rotation. Therefore, wherever we mention moment of inertia, the axis of rotation must be specified. In the present case I is the moment of inertia about an axis passing through the point C (Fig 7.9).

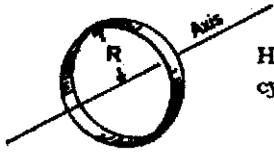
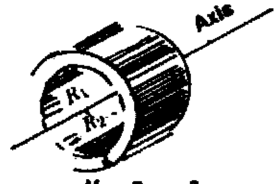
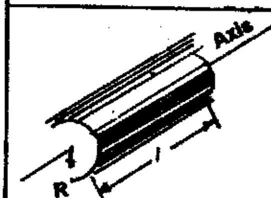
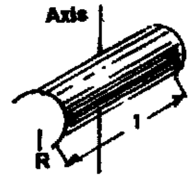
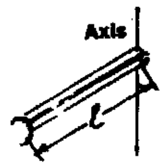
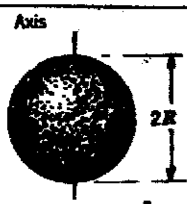
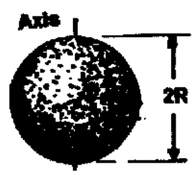
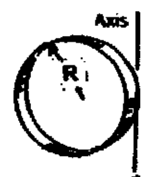
The unit of moment of inertia is obvious from its definition. It is kg m^2.

The moment of inertia of a rigid body is often written as

$$I = MK^2 \qquad (7.15)$$

where M is the total mass of the body and K is called the **radius of gyration** of the body. ***The radius of gyration is that distance from the axis of rotation where the whole mass of the body can be placed to get the same moment of inertia which the body actually has.*** There is certain advantage in writing moment of inertia in this form. If the body is regular and has the same properties every where (such a body is called homogeneous), then K can be computed from its geometry and the moment of inertia can be easily calculated. Table 7.2 shows the moments of inertia of a few regular bodies.

Table 7.2 : *Moment of Inertia of some regular bodies*

 Hoop about cylinder axis $I = MR^2$	 Annular cylinder (or ring) about cylinder axis $I = \frac{M}{2}(R_1^2 + R_2^2)$
 Solid cylinder about cylinder axis $I = \frac{MR^2}{2}$	 Solid cylinder (or disk) about a central diameter $I = \frac{MR^2}{4} + \frac{Ml^2}{12}$
 Thin rod about axis through centre ⊥ to length $I = \frac{Ml^2}{12}$	 Thin rod about axis through one end ⊥ to length $I = \frac{Ml^2}{3}$
 Solid sphere about any diameter $I = \frac{2MR^2}{5}$	 Thin spherical shell about any diameter $I = \frac{2MR^2}{3}$
 Hoop about any diameter $I = \frac{MR^2}{2}$	 Hoop about any tangent line $I = \frac{3MR^2}{2}$

Look carefully at equation (7.13). Compare this with the kinetic energy of a body in linear motion. What do you find? In the rotational motion the role of mass has been taken over by the moment of inertia and the angular velocity has replaced the linear velocity.

Physical significance of M.I.

The physical significance of the moment of inertia is that it performs the same role in the rotational motion that the mass does in linear motion.

Just as the mass of a body resists change in its state of linear motion, the *moment of inertia resists a change in its rotational motion.* This property of the moment of inertia has been put to a great practical use. Most machines which produce rotational motion have as one of their components a disc which has a very large moment of inertia. Examples of such machines are the steam engine and the automobile engine. The disc with a large moment of inertia is called a ***flywheel***. To understand how a flywheel works, imagine that the driver of the engine wants to increase suddenly the speed. Because of its large moment of inertia, the flywheel resists this attempt. It allows only a gradual increase in speed. Similarly, it works against the attempts to suddenly reduce the speed, and allows only a slow decrease in the speed. Thus, the flywheel, with its large moment of inertia, prevents jerky motions and ensures a smooth ride for the passangers.

We have noted that in rotational motion the angular velocity corresponds to the linear velocity in linear motion. Since angular acceleration (denoted usually by α) is the rate of change of angular velocity, it must corresponds to the linear acceleration in the linear motion. ***Use similar argument to show that the angle θ through which a body rotates corresponds to the distance covered in linear motion.***

We can now write down relations for the rotational motion similar to the ones you have derived for the linear motion.

$$\theta = \omega_0 t + \tfrac{1}{2} \alpha t^2 \qquad (7.16)$$

We have mentioned above that for the rotational motion of a rigid body its CM is kept fixed. It is not necessary that CM be kept fixed, it is convenient to do so. Any other point can be fixed and the body can still have rotational motion. But now the axis of rotation will pass through this fixed point. The moment of inertia about this axis would be different from the moment of inertia about an axis passing through the CM. What is the relation between these two? Let us find out.

7.5.2 Theorems of moment of inertia

There are two theorems which connect moments of inertia about various axes. These are

(i) ***the theorem of parallel axes,*** and
(ii) ***the theorem of perpendicular axes.***

We explain below these theorems and their applications.

Suppose the given rigid body rotates about an axis passing through any point P other than the centre of mass. The moment of inertia about this axis can be found from a knowledge of the moment of inertia about a parallel axis through the centre of mass. If I denotes the required moment of inertia and I_c denotes the moment of inertia about a parallel axis through the CM, then

$$I = I_c + Md^2 \qquad (7.17)$$

where M is the mass of the body and d is the distance between the two axes (Fig 7.10). This is known as the **theorem of parallel axes.**

The other useful theorem concerning the moment of inertial is known ***the theorem of perpendicular axes.*** It is applicable only to plane bodies. Let us choose three mutually perpendicular axes, two of which, say x and y are in the plane of the body, and the third, the z axis, is perpendicular to the plane (Fig 7.11). The theorem states that ***the sum of the moments of inertia about axes x and y is equal to the moment of inertia about the z axis.***

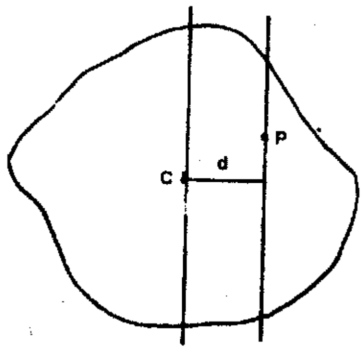


Fig 7.10 : *Theorem of parallel axes.*

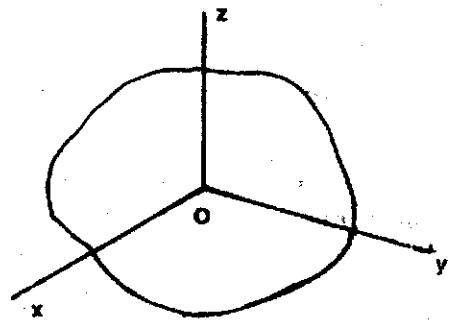


Fig 7.11: *Theorem of perpendicular axes.*

Thus,

$$I_z = I_x + I_y \qquad (7.18)$$

We illustrate the use of these theorems by the following example.

For example let us take a hoop shown in Fig 7.12. Table 7.2 shows that its moment of inertia about the cylinder axis is MR^2, where M is its mass and R its radius. The theorem axes tells us that this must be equal to the sum of the moments of inertia about two diameters which are perpendicular to each other. Now the symmetry of the hoop tells us that the moment of inertia about any diameter is the same as about any other diameter. This means that all the diameters are **equivalent** and any two perpendicular diameters may be chosen. Since the moment of inertia about each is the same, say I_d, (7.18) gives

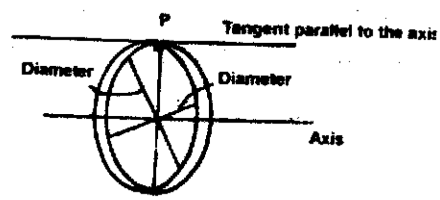


Fig 7.12 : *Moment of inertia of a hoop.*

$$MR^2 = 2I_d$$

and therefore

$$I_d = \tfrac{1}{2}MR^2 .$$

Let us now take a point P on the rim. Consider a tangent to the hoop at this point which is parallel to the axis of the hoop. The distance between the two axes is obviously equal to R. The moment of inertia about the tangent is found by the application of the theorem of parallel axes. It is given by

$$I_{tan} = MR^2 + MR^2 = 2MR^2.$$

It must be noted that many of the entries in Table 7.2 have been computed using the theorems of parallel and perpendicular axes.

7.5.3 Torque and Couple

Have you ever noticed that we always open the door by applying force at a point far from the hinges? What happens if you try to open a door by applying force near the hinges? Carry out this little activity. You would realize that much more effort is needed to open the door if you apply force near the hinges. Why is it so?

You would also have noticed that for turning a screw we use a spanner with a long handle. What is the advantage of a long handle?

Let us seek answers to these questions.

Suppose O is a point fixed in the body so that body can rotate about an axis through this point (Fig 7.13). Let a force of magnitude F be applied at the point B along the line AB. If AB passes through the point O, the force F will not be able to rotate the body. The farther is the line AB from O, the greater is the ability of the force to turn the body about the axis through O. The ***turning effect of a force is called a torque***. Its magnitude is given by

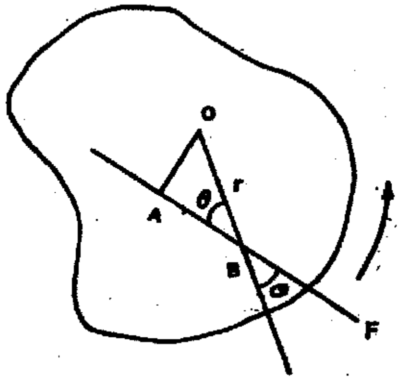


Fig 7.13 : *Rotation of a body.*

$$\tau = Fa = Fr \sin \theta \tag{7.19}$$

The units of torque are Newton-metre, or Nm. The torque is actually a vector quantity. The Vector form of (7.19) is

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} \tag{7.20}$$

which gives both the magnitude and the direction of the torque. What is the direction in which the body would turn? To find this we notice that by the rules of vector product (refer to lesson 1), $\boldsymbol{\tau}$ is perpendicular to the plane containing vectors $\boldsymbol{r}$ and $\boldsymbol{F}$ (Fig 7.14). If we extend the thumb of the right hand in the direction of τ, then the direction of turning of the body is given by the sense in which the fingers are curled. *Apply the above rule and show that the turning effect of the force in Fig 7.13 is in the anticlockwise direction.*

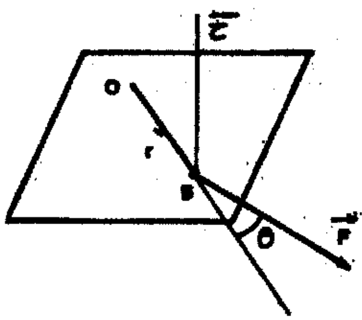


Fig 7.14 : *Right hand thumb rule*

If there are several torques acting on a body then the net torque acting on the body is the sum of all the torques. Do you see any correspondence between the role of torque in the rotational motion and the role of force in the linear motion? We will see the correspondence better a little later. For the moment suppose that there are two forces of equal magnitude acting on the body in opposite directions (Fig 7.15). The two torques on the body have magnitudes

$$\tau_1 = (a + b)F$$

$$\tau_2 = aF.$$

The turning effects of the two torques are in the opposite directions.

Therefore, the net turning effect on the body is

$$\tau = \tau_1 - \tau_2 = bF \qquad (7.21)$$

in the direction of the larger torque, which in this case is τ_1.

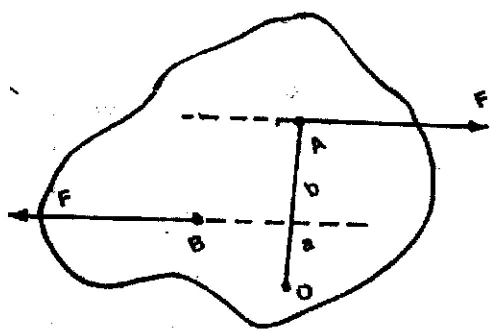


Fig 7.15 : *Two opposite forces acting on a body.*

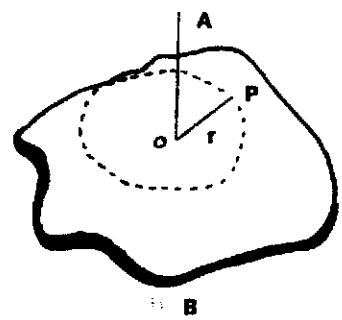


Fig 7.16 : *A rigid body rotating about on axis.*

Two equal and opposite forces are said to form a couple whose torque is equal to the product of one of the forces and the perpendicular distance between them.

There is another useful expression for torque which makes the correspondence between it and force in the linear motion quite clear. Consider a rigid body rotating about an axis through a point O (Fig 7.16). Obviously a particle like P is rotating about the axis in a circle of radius r. If the circular motion is nonuniform, then the particle experiences forces in the radial direction as well as in the tangential direction. The radial force is the familiar centripetal force $m\omega^2 r$ which keeps the particle in the circular path. The tangential force is required to change the magnitude of v, which at every instant is in the direction of the tangent to the circular path. Its magnitude is ma, where a is the tangential acceleration. ***The radial force does not produce any torque.*** *Can you say why? The tangential force produces a torque of magnitude mar.* Since $a = r\alpha$, where α is the angular acceleration, the magnitude of the torque is $mr^2\alpha$. If we consider all the particles of the body then

$$\tau = \left(\sum_i m_i r_i^2 \right) \alpha$$

$$= I\alpha . \qquad (7.22)$$

The similarity between this equation and $F = ma$ shows that τ performs the same role in rotational motion as F does in linear motion. A list of corresponding quantities in rotational motion and linear motion is given in Table 7.3. With the help of this table, we can write down any equation for rotational motion if we know its corresponding equation in linear motion.

Table 7.3 : *Corresponding quantities in rotational and linear motion.*

Rectilinear Motion		Rotation about a Fixed Axis	
Displacement	x	Angular displacement	θ
Velocity	$v = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Mass	M	Rotational inertia	I
Force	$F = ma$	Torque	$\tau = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau d\theta$
Kinetic energy	$\frac{1}{2}Mv^2$	Kinetic energy	$\frac{1}{2}I\omega^2$
Power	$P = Fv$	Power	$P = \tau\omega$
Linear momentum	Mv	Angular momentum	$I\omega$

With the help of (7.22) we can calculate the angular acceleration produced in body by a given torque.

Example 7.2: *A uniform disc of mass 1.0 kg and radius 0.1 m can rotate about an axle without friction. A massless string goes round the rim (Fig 7.17). If a pull of 1.0 kg is applied to the string, find the angular acceleration of the disc, the angle through which the disc rotates in one second, and the angular velocity of the disc after one second.*

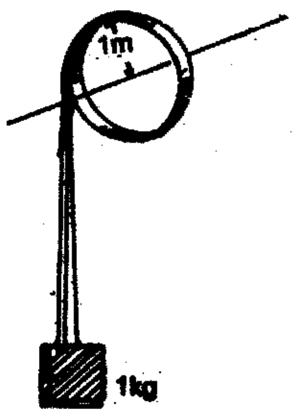


Fig 7.17

Solution: If R and M denote the radius and mass of the disc, then according to Table 7.2, $I = \frac{1}{2}MR^2$. If P denotes the pull then $\tau = PR$. Eq. (7.22) now gives

$$\alpha = \tau/I = PR/I = 2P/MR$$

$$\frac{2 \times 1.0}{1.0 \times 0.1} = 20 \text{ rad}/\text{sec}^2$$

For the angle θ through which the disc rotates, we use Eq 7.17. Since the initial angular velocity is zero, we have

$$\theta = \frac{1}{2} \times 20 \times 1.0 = 10 \text{ rad}$$

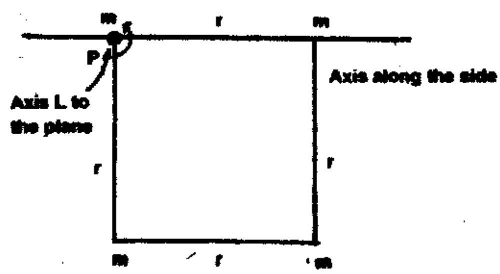
For the velocity after one second, we use Eq 7.16. We have

$$\omega = \alpha t = 20 \times 1.0 = 20 \text{ rad/sec}.$$

Now, let us check your progress.

INTEXT QUESTIONS 7.3

1. Four particles, each of mass m, are fixed at the corners of a square whose each side is of length r. Find the moment of inertia about an axis passing through one of the corners and perpendicular to the plane of the square. Calculate also the moment of inertia about an axis which is along one of the sides. Verify your result by using the theorem of perpendicular axes.



....

2. Find the radius of gyration of a solid sphere if the axis is a tangent to the sphere. (You will have to use Table 7.2.)

....

3. Calculate the moment of inertia of the system shown in question number 1 of Intext Questions 7.2 if the axis of rotation passes through O and is perpendicular to the plane of the paper. From this find the moment of inertia about a parallel axis through the centre of mass.

....

4. Ten particles, each of mass m, are placed at distance r each from the axis of rotation, as shown.

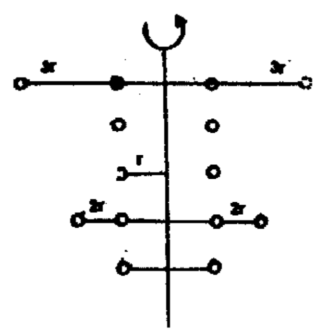
a) Calculate the moment of inertia of the system.

....

b) Suppose the distances of particles 4 and 9 are changed to $3r$ each. Find the moment of inertia now and compare it with that found in (a).

....

c) Particles 2 and 7 are displaced to distances $2r$ each, and particles 1 and 6 are displaced to distances $3r$ each. Calculate once again the moment of inertia and compare with those found in (a) and (b).



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7.6 ANGULAR MOMENTUM

Have you ever seen a toy umbrella floating in air with its direction fixed and wondered how it is able to maintain a fixed direction?

If you can get hold of a stool which can rotate without much friction, you can perform an interesting experiment. Ask a friend of yours to sit on the stool with her arms folded. Make the stool rotate fast and then let it go. Ask your friend to stretch her arms and notice if there is any change in the speed of rotation of the stool. Ask her to fold her arms once again and observe change in the speed of the stool.

Let us try to understand why we expected a change in the speed of rotation of the stool in the last experiment. For this consider once again a rigid body rotating about an axis, say the axis z through a fixed point O in the body. All the points of the body describe circular paths about the axis with the centres of the paths on the axis. Consider a particle like P at a distance r_i from the axis (Fig 7.18). Its linear velocity is $r_i\omega$ and its momentum is therefore $mr_i\omega$. ***The product of linear momentum and the distance from the axis***

is called angular momentum, denoted by L. If we sum this product for all the particles of the body, we get

$$L = \left(\sum_i m_i r_i^2 \right) \omega$$

$$= I\omega \qquad (7.23)$$

Remember that the angular velocity is the same for all the particles and the product within brackets is the moment of inertia. Like the linear momentum, the angular momentum is also a vector quantity. Eq. (7.23) gives only the component of the vector $\boldsymbol{L}$ along the axis of rotation*. It is important to remember that I must refer to the same axis. The units of angular momentum are kg-m²/sec.

*In general L is not parallel o the angular velocity ector. However, in a rigid ody we can find three utually perpendicular axes assing through the centre f mass. Along any one of ese axes the components f angular momentum and gular velocity are parallel q (7.23) has been written ith this in mind.

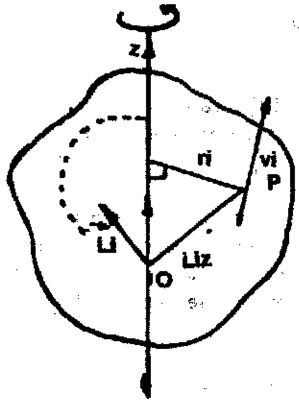


Fig 7.18 : *A rigid body rotating about an axis.*

Recall now that the rate of change of ω is α and $I\alpha = \tau$. Therefore, ***the rate of change of angular momentum is equal to the torque***. Or, using the full vector notation,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{\tau} = I\frac{d\boldsymbol{\omega}}{dt} = I\boldsymbol{\alpha} \qquad (7.25)$$

which is the equation of motion of a rotating body.

7.6.1 Conservation of angular momentum

Eq (7.24) shows that ***if there is no net torque acting on the body then the angular momentum stays constant. This is the principle of conservation of angular momentum.*** Along with the conservation of energy and linear momentum, this is one of the most important principles of physics.

The principle of conservation of angular momentum allows us to answer the questions raised in the beginning of Section 5. In the case of the toy umbrella the trick is to make it rotate and thereby impart it some angular momentum. Once it is let go in air, there is no torque acting on it. Its angular momentum is then constant. Since angular momentum is a vector quantity, its constancy implies fixed direction and magnitude. Thus, the direction of the toy umbrella remains fixed while it is in air.

In the case of your friend on the rotating stool, when there is no torque acting on the stool the angular momentum of the stool and the person on it must be conserved. When the person stretches her arms, she causes the moment of inertia of the system to increase. Eq (7.23) implies then that the angular velocity must decrease. Similarly, when she folds her arms, the moment of inertia of the system is decreased. This causes the angular velocity to increase. (It might help you to look at question number 4 of Intext Questions 7.4). *When your friend stretches her arms why does the moment of inertia of the system (the person and the stool) increase?*

Let us look at a few more examples of the conservation of momentum. Suppose we have a spherical ball of mass M and radius R. The ball is set rotating by applying a torque to it. The torque is then removed. Since there is no external torque now, whatever angular momentum the ball has acquired must be conserved. The moment of inertia of the ball is 2/5 MR^2 (Table 7.2). Therefore, it has angular momentum given by

$$L = \frac{2}{5} MR^2 \omega \qquad (7.25)$$

where ω is its angular velocity. Imagine now that the radius of the ball somehow decreases. To conserve its angular momentum, the ball must rotate faster. This is what really happens to some stars, such as those which becomes pulsars (see Box on page 147). *What would happen if the radius of the ball were to increase suddenly?*

Acrobats, skaters, divers and other sports persons make excellent use of the principle of conservation of angular momenm to show off their feats. You must have seen on the TV divers jumping off the diving boards during swimming events in national or international tournaments. At the time of jumping the diver gives herself a slight rotation, by which she acquires some angular momentum. When she is in air, there is no torque acting on her and therefore her angular momentum must be conserved. If she folds her body to decrease her moment of inertia (Fig 7.19) her rotation must become faster. If she unfolds her body then her moment of inertia increases and she must rotate slower. In this way, by controlling the shape of her body, the diver is able to show off her feat before falling into water.

Fig 7.19 : *Divers jumping off the diving boards.*

INTEXT QUESTIONS 7.4

1. A molecule consists of two identical atoms of mass m each and distance d apart. This distance remains fixed. The molecule rotates about an axis which is halfway between the two atoms with angular speed ω. Calculate the angular momentum of the molecule.

....

2. A uniform circular disc of mass 2.0 kg and radius 20 cm is rotated about one of its diameters at an angular speed of 10 rad/sec. Find its angular momentum about the axis of rotation.

....

3. A wheel is rotating at an angular speed ω about its axis which is kept vertical. Another wheel of the same radius but half the mass, initially at rest, is slipped on the same axle gently. The two wheels then rotate with the same speed. Find the common angular speed

....

4. It is said that the earth was formed from a contracting gas cloud. Suppose some time in the past the radius of the earth was 2.5 times its present radius. What was then its period of rotation on its own axis?

....

7.7 MOTION ON AN INCLINED PLANE

We have already noted that if a point in a rigid body is not fixed, then it can possess rotational motion as well translational motion (Remember the slow motion picture of a spinning ball?). The general motion of a rigid body consists of both these motions. As an interesting example of the combined motion, let us consider the motion of cylindrical and spherical bodies on an inclined plane or a slope.

Suppose the rigid body has mass M, radius R and moment of inertia I. It is moving down an inclined plane of height h (Fig 7.20). At the end of the journey it has acquired a linear velocity v and an angular velocity ω. We assume that the loss of energy due to friction is small and can be neglected. The principle of conservation of energy then implies that the sum of the kinetic energies due to translation and rotation must be equal to the potential energy that the body had at the top of the inclined plane. Therefore,

$$\tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2 = Mgh \qquad (7.26)$$

If there is no slipping, then $v = R\omega$.

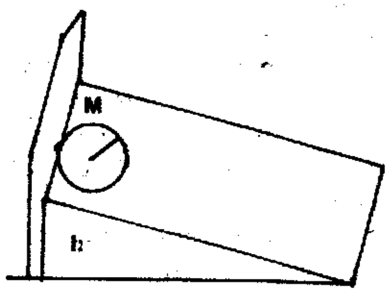


Fig 7.20 : Motion of a rigid body on an inclined plane.

Then we get, $$\frac{1}{2}Mv^2 + \frac{1}{2}I\frac{v^2}{R^2} = Mgh \qquad (7.27)$$

To take a simple example, let the body be a hoop. Table 7.2 shows that its moment of inertia about its own axis is MR^2. Eq. (7.27) then gives that

$$v = \sqrt{gh} \qquad (7.28)$$

Do you notice any thing interesting in this equation? ***The linear velocity has turned out to be independent of the mass and the radius of the hoop. So, a hoop of any material and any radius rolls with the same speed on the inclined plane.***

7.5 INTEXT QUESTIONS

1. A solid sphere rolls down a slope without slipping. What will be its velocity in terms of the height of the slope?

   ....

2. A solid cylinder rolls down an inclined plane witnout slipping. What fraction of its kinetic energy is translational? What is the magnitude of its velocity after falling through a height h?

   ....

3. A uniform sphere of mass 2 kg and radius 10 cm is released from rest on an inclined plane which makes 30° angle with the horizontal. Deduce its (a) angular accleration, (b) linear accleration along the plane, and (c) kinetic energy as it travels 2 m along the plane.

   ....

SECRET OF PULSARS

An interesting example of the conservation of angular momentum is provided by stars. You must have heard of pulsars. These are stars which send towards us pulses of radiation of great intensity. The pulses are periodic and the periodicity is extremely precise. The time periods range between a few milliseconds to a few seconds. It is believed that such short time periods show that the stars are rotating very fast. Most of the matter of these stars is in the form of neutrons. The neutrons, as you might be knowing, along with the protons are the building blocks of the atomic nuclei. These stars are called neutron stars. These stars represent the last stage in the life of some stars. The secret of their fast rotation is their tiny size. The radius of a typical neutron star is only 10 km. Compare this with the radius of the sun, which is about 7×10^5 km. The sun rotates on its axis with a period of about 25 days. Imagine that the sun suddenly shrinks to the size of a neutron star without any change in its mass. You can show that in order to conserve its angular momentum the sun has to rotate with a period as short as the fraction of a millisecond.

7.8 WHAT YOU HAVE LEARNT

- A rigid body has rotational as well as translational motion
- The equation of translational motion may be written in the same form as for a single particle
- If a point in the rigid body is fixed then it can only rotate
- The moment of inertia about an axis of rotation is defined as $\Sigma m_i r_i^2$
- The moment of inertia plays the same role in rotational motion as the mass does in linear motion
- That the turning effect of a force $\boldsymbol{F}$ on a rigid body is given by the torque τ = $\times$ $\boldsymbol{F}$
- Two equal and opposite forces constitute a couple, the magnitude of whose turning effect is equal to the product of one of the forces and the distance between the forces
- The application of an external torque changes the angular momentum of the body
- If the sum of all external torques is zero then the angular momentum of the body remains constant
- When a cylindrical or a spherical body rolls down an inclined plane without slipping, its speed is independent of its mass and radius.

7.9 TERMINAL QUESTIONS

1. The weight Mg of a body is shown generally as acting at the centre of mass of the body. Does this mean that the earth does not attract other particles?

2. Is it possible for the centre of mass of a body to lie outside the body? Give two examples to justify your answer?

3. Manuals for car-engine requires always specify the torque to be when tightening the cylinder head bolts. Is it the right thing to do? Justify your answer.

4. In a molecule of carbon monoxide (CO) the nuclei of the two atoms are 1.13×10^{-10} m apart. Find the location of the centre of mass of the molecule.

5. A grinding wheel of mass 5.0 kg and diameter 0.20 m is rotating with an angular velocity of 490 rad/sec. Find its kinetic energy. Through what distance would it have to drop in free fall to acquire this kinetic energy?

6. Two identical spheres, each of mass 1.0 kg and radius 0.10 m, are attached at the end of a massless rod. The distance between the two balls is 0.50 m. Calculate the moment of inertia of the system about an axis passing through the midpoint of the rod and perpendicular to it.

7. A wheel of diameter 1.0 m is rotating about a fixed axis with an initial angular velocity of 2 rev/sec. The angular acceleration is 3 rev/sec^2.

a) Compute the angular velocity after 2 seconds.
b) Through what angle has the wheel turned in this time?
c) What is the tangential velocity of a point on the rim of the wheel at $t = 2$ sec?
d) What is the resultant acceleration of a point on the rim of the wheel at $t = 2$ sec?

8. A wheel rotating at an angular speed of 20 rad/sec is brought to rest by a constant torque in 4.0 seconds. If the moment of inertia of the wheel about the axis of rotation is 0.20 kg-m^2, find the work done by the torque in the first two seconds.

9. Two wheels are mounted on the same axle. Wheel A has moment of inertia 5×10^{-2} kg-m^2 and wheel B has moment of inertia 0.2 kg-m^2. Wheel A is set spinning at 600 rev/min, while wheel B is stationary. A clutch now acts to join A and B so that they must spin together.
(a) At what speed will they rotate?
(b) Suppose the clutch acts gradually. Will the end result be the same as though they were joined suddenly? (Neglect any friction at the bearings).
(c) How does the rotational kinetic energy before joining compare with the kinetic energy after joining?
(d) What torque does the clutch deliver if A makes 10 revolutions during the operation of the clutch?

10. A solid sphere of mass m rolls down a slope without slipping. Show that the force of friction acting on it is $(2/7)\ mg \sin\theta$, where θ is the inclination of the slope.

11. You are given two identically looking spheres and told that one of them is hollow. Suggest a method to detect the hollow one.

12. The moment of inertia of a wheel is 1000 kg-m^2. Its rotation is uniformly accelerated. At some instant of time its angular velocity is 10 rad/sec. After the wheel has rotated through an angle of 100 radians, the angular velocity of the wheel becomes 100 rad/sec. Calculate the torque applied to the wheel and the change in its kinetic energy.

13. A disc of radius 10 cm and mass 1 kg is rotating about its own axis. It is accelerated uniformly from rest. During the first second it rotates through 2.5 radians. Find the angle rotated during the next second. What is the magnitude of the torque acting on the disc?

ANSWERS TO THE INTEXT QUESTIONS

Intext Questions 7.1

1. Yes.

Intext Questions 7.2

1. The coordinates of given five masses are A (–2, –2), B (–5, –1), C (7, 3), D (2, 6) and E (0, 3) and their masses are 1 kg, 2 kg, 3 kg, 4 kg and 5 kg respectively.
Hence, coordinates of centre of mass of the system are

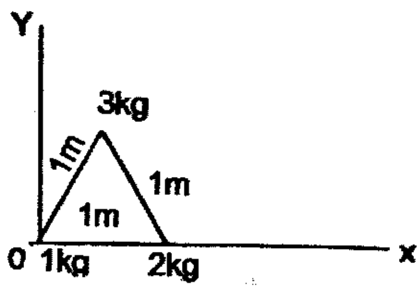
$$x = \frac{-2\times1+5\times2+7\times3+2\times4+0\times5}{1+2+3+4+5}$$

$$= \frac{37}{15} = 2.47 \text{ and}$$

$$y = \frac{-2\times1+(-1)\times2+3\times3+6\times4+3\times5}{1+2+3+4+5}$$

$$= \frac{44}{15} = 2.97$$

2. Let the three particle system is as shown in Fig.



Hence, Centre of mass,

$$x = \frac{1\times0+2\times1+3\times.5}{1+2+3} = \frac{3.5}{6}\text{ m}$$

$$y = \frac{1\times0+2\times0+3\times\left(\sqrt{3}/2\right)}{1+2+3}$$

$$= \frac{\sqrt{3}}{4}\text{ m}$$

Centre of Mass is $\left(\frac{3.5}{6}, \frac{\sqrt{3}}{4}\right)$

Intext Questions 7.3

1. Moment of inertia of the system about an axis perpendicular to the plane passing point P,
$= mr^2 + m(2r^2) + mr^2 = 4\,mr^2$
M.I. about the axis along the side
$= mr^2 + mr^2 = 2\,mr^2$

2. M.I. of a solid sphere about an axis tangential to the sphere
$$= \frac{7}{5} MR^2.$$
Which should be equal to MK^2
$$\therefore K^2 = \frac{7}{5} R^2$$
or Radius of gyration $K = R\sqrt{\frac{7}{5}}$

3. M.I. of the system about an axis perpendicular to the plane and passing through 0 is
$= 1\times OA^2+2\times OB^2+3\times OC^2+4\times OD^2+5\times OE^2$
$= 1\times8+2\times26+3\times58+4\times40+5\times9$
$= 439$ kg m².

4. (a) M.I. of the system $= 10\,mr^2$
(b) M.I. $= 26\,mr^2$
(c) M.I. $= 30\,mr^2$

Intext Questions 7.4

1. Angular momentum
$$= \left(m\frac{d^2}{4} + m\frac{d^2}{4}\right)\omega$$
$$L = \frac{md^2\omega}{2}$$

2. Angular momentum about an axis of rotation (diameter).
$$L = I\omega = m\frac{r^2}{4}\times\omega$$
$$= 2\times\frac{(.2)^2}{4}\times 10 = 0.2 \text{ kg m}$$

3. Accord: g to conservation of angular momentum
$$I\omega^2 = (I_1 + I_2)\,\omega_1^2$$
$$mr^2\,\omega^2 = \left(mr^2 + \frac{m}{2}r^2\right)\omega_1^2$$
$$\omega^2 = \frac{3}{2}\,\omega_1^2$$
$\therefore$ Common angular speed $\omega_1 = \omega\sqrt{2/3}$

4. Let the present period of revolution of earth is T. According to the conservation of angular momentum,
$$\frac{2}{5}M(2.5R)^2 \times \left(\frac{2\pi}{T_0}\right)^2$$
$$= \frac{2}{5}MR^2 \times \left(\frac{2\pi}{T}\right)^2$$
It gives, $T_0^2 = 6.25\,T^2 \quad \therefore T_0 = 2.5\,T$
Thus, period of revolution of earth in the past is $T_0 = 2.5$ times the present time period.

Intext Questions 7.5

1. Using equation 7.27, for a solid sphere
$$\frac{1}{2}mv^2 + \frac{1}{2}\times\frac{2}{5}mr^2 \cdot \frac{v^2}{r^2} = mgh.$$
It gives $v = \sqrt{\frac{10}{7}.gh}$

2. For a solid cylinder, $I = \frac{mR^2}{2}$
$\therefore$ Total K.E
$$= \frac{1}{2}mv^2 + \frac{1}{2}\frac{mR^2}{2}\cdot\frac{v^2}{R^2} = \frac{3}{4}mv^2$$
Hence, fraction of translational K.E.
$$= \frac{\frac{1}{2}mv^2}{\frac{3}{4}mv^2} = \frac{2}{3}$$

3. Linear acceleration along the plane
$$a = \frac{g\sin\theta}{1 + K^2/R^2} = \frac{9.8\times\frac{1}{2}}{1+\frac{2}{5}} = 3.5 \text{ ms}^{-2},$$
$$\left[\therefore \text{for sphere } K^2 = \frac{2}{5}R^2\right]$$
Angular acceleration $= \frac{a}{R} = \frac{3.5}{0.1}$
$= 35$ rad s⁻²

Gain in K.E. = loss in P.E. in vertical descent h ($= l\sin\theta$)

$= Mgl\sin\theta = 2\times9.8\times2\times\frac{1}{2} = 19.6$ J.